Proof of Kac and Rudakov's Conjecture on Generalized Verma Module over Lie Superalgebra $\mathbf{E}(5,10)$*

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Abstract

The exceptional infinite-dimensional linearly compact simple Lie superalgebra $\mathrm{E}(5,10)$, which Kac believes, is the algebra of symmetries of the SU_5 Grand Unified Model. In this paper, we give a proof of Kac and Rudakov's conjecture about the classification of all the degenerate generalized Verma module over $\mathrm{E}(5,10)$. Also, we work out all the nontrivial singular vectors degree by degree. It is a potential that the representation theory of $\mathrm{E}(5,10)$ will shed new light on various features of the the SU_5 Grand unified model.

1 Introduction

A linearly compact infinite-dimensional Lie algebra is a topological Lie algebra whose underlying space is a topological space isomorphic to the space of formal power series over complex field in finite number of variables with formal topology. Cartan's list of linearly compact infinite-dimensional simple Lie algebras consists of four series: the Lie algebra of all complex vector fields and its subalgebras of divergence 0 vector fields, symplectic vector fields and contact vector fields.

Kac proved the "super" version of this result. In other words, he classified linearly compact infinite-dimensional Lie superalgebras [K1]. There turn out to be 10 families and 5 exceptions, which are called $\mathrm{E}(1,6), \mathrm{E}(3,6), \mathrm{E}(3,8), \mathrm{E}(4,4)$ and $\mathrm{E}(5,10)$. Many of the families are straightforward "super" generalizations of the 4 families of linearly compact infinite-dimensional simple Lie algebras. Some are stranger. Most important for us today are the 5 exceptions discovered by Irina Shchepochkina [Sh].

The representation theory of $E(3,6)$ and $E(3,8)$ was developed by Kac and Rudakov [KR1-KR3], and some further observations were made on its connections to the Standard Model [K2]. It was found quite remarkable that the SU_5 Grand unified model of Georgi-Glashow combines the left multiplets of fundamental fermions in precisely the negative part of the consistent gradation of $\mathrm{E}(5,10)$. This is perhaps an indication of the possibility that an extension from su_5 to algebra of internal symmetries may resolve the difficulties with the proton decay. It is a potential that the representation theory of $\mathrm{E}(5,10)$ will shed new light on various features of the the SU_5 Grand unified model.

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As to the representation theory of $E(5,10)$, Kac and Rudakov formulate an conjecture [KR3], which can be stated as follows. The Lie superalgebra $L = E(5,10)$ carries a unique consistent irreducible $\mathbb{Z}$-gradation $L = \bigoplus_{j\geq -2} L_j$, where L_0 is isomorphic to simple Lie algebra sl_5. Given L_0- module V, we extend it to a L module by letting L_+ acts trivially, and define the induced module

$$M(V) = U(L) \otimes_{U(L_0)} V \cong U(L_-)V.$$

If V is finite-dimensional irreducible L_0 -module, the L module $M(V)$ is called a *generalized Verma module* associated to V, and it is called *degenerate* if it is not irreducible.

We denote by $\mathrm{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ the finite-dimensional irreducible L_0 -module with highest weight $\sum_{i=1}^{4} \lambda_i \omega_i$, where $\omega_1, \omega_2, \omega_3, \omega$ are the fundamental weights for sl_5. Let

$$M = M(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = M(V(\lambda_1, \lambda_2, \lambda_3, \lambda_4))$$

denote the corresponding generalized Verma module over $\mathrm{E}(5,10)$. Denote by $\mathbb{N}$ the additive semigroup of nonnegative integers.

Kac and Rudakov's Conjecture *The following is a complete list of degenerate Verma modules over* $E(5,10)$*:*

$$M(m,n,0,0); M(m,0,0,n); M(0,0,m,n) \ (m,n \in \mathbb{N}).$$

In this paper, we give a proof of this conjecture and work out all the nontrivial singular vectors for any generalized Verma module over $\mathrm{E}(5,10)$. The first key point of our proof is investigating that there exists a grading on the generalized Verma module, through which we find that any singular vector is controlled by its leading term via an exponential-like differential operator, where the leading term lies in certain tensor product module of sl_5; the second one is the observation of an irreducible tensor operator of rank $\omega_1 + \omega_2$ for simple Lie algebra sl_5, which plays the center role in our calculation of all the singular vectors.

The paper is organized as follows: In section 2, we recall Kac's geometric construction of Lie superalgebra $\mathrm{E}(5,10)$ and the KR conjecture. In Section 3, we provide some techniques concerning the irreducible tensor operators and tensor module decomposition theory of simple Lie algebra. In Section 4, we prove that all the nontrivial singular vectors are of degree less than or equal to four. Also, the leading term of any singular vector must lie in one of the tensor decomposition of four tensor product module of sl_5 (cf. Theorem 4.6). In Section 5, we work out all the nontrivial singular vectors degree by degree (cf. Theorem 5.3, Theorem 5.4, Theorem 5.5, Theorem 5.6).

2 Lie superalgebra $\mathrm{E}(5,10)$ and KR conjecture

In this section, we recall Kac's geometric construction of Lie superalgebra $\mathrm{E}(5,10)$ and KR Conjecture which are stated in [KR3].

For two integers $m<n$, we denote $\overline{m,n}=\{m,m+1,\cdots,n\}$. Let

$$\mathrm{W}_n=\{\sum_{i=1}^{n}p_i(x)\partial_i\mid p_i(x)\in\mathbb{C}[[x_1,\cdots,x_n]],\partial_i=\partial_{x_i}\}\tag{2.1}$$

denote the Lie algebra of formal vector fields in n indeterminates;

$$\mathrm{S}_n=\{D=\sum_{i=1}^{n}p_i\partial_i\mid \mathrm{div}D=\sum_{i=1}^{n}\partial_i(p_i)=0\}\tag{2.2}$$

denote the Lie subalgebra of divergenceless formal vector fields; $\Omega^k(n)$ denote the associative algebra of formal differential forms of degree k in n indeterminates, $\Omega^k_{\mathrm{cl}}(n)$ denote the subspace of closed forms.

The exceptional infinite-dimensional linearly compact Lie superalgebra $\mathrm{E}(5,10)=\mathrm{E}(5,10)_{\underline{0}}+\mathrm{E}(5,10)_{\underline{1}}$ is constructed as follows:

$$\mathrm{E}(5,10)_{\underline{0}}=S_5,\ \ \mathrm{E}(5,10)_{\underline{1}}=\Omega^2_{\mathrm{cl}}(5),\tag{2.3}$$

where $\mathrm{E}(5,10)_{\underline{0}}$ acts on $\mathrm{E}(5,10)_{\underline{1}}$ via the Lie derivative,

$$[\omega_2,\omega_2']=\omega_2\wedge\omega_2'\in\Omega^4_{\mathrm{cl}}(5)=S_5\tag{2.4}$$

for $\omega_2,\omega_2'\in\mathrm{E}(5,10)_{\underline{1}}$.

We use for the odd elements of $\mathrm{E}(5,10)$ the notation $d_{ij}=dx_i\wedge dx_j(i,j\in\overline{1,5})$; recall that we have the following commutation relation ($f,g\in C[[x_1,\cdots,x_5]]$):

$$[fd_{jk};gd_{lm}]=\varepsilon_{ijklm}fg\partial_i;\tag{2.5}$$

where

$$\varepsilon_{ijklm}=\begin{cases}\text{the sign of the permutation (ijklm)}, & \text{if all indices } ijklm \text{ are distinct,}\\ 0, & \text{otherwise.}\end{cases}\tag{2.6}$$

And the Lie superalgebra $\mathrm{L}=\mathrm{E}(5,10)$ carries a unique consistent irreducible $\mathbb{Z}$-gradation $\mathrm{L}=\bigoplus_{j\geq-2}\mathrm{L}_j$. It is defined by:

$$degx_i=2=-\partial_i,degd_{ij}=-1\tag{2.7}$$

One has: $\mathrm{L}_0\simeq\mathrm{sl}_5$ and the L_0-modules occurring in the negative part are:

$$\mathrm{L}_{-1}=\mathrm{Span}_{\mathbb{C}}\{d_{ij}\mid i,j\in\overline{1,5}\}\simeq\Lambda^2\mathbb{C}^5,$$

$$\mathrm{L}_{-2}=\mathrm{Span}_{\mathbb{C}}\{\partial_i\mid i\in\overline{1,5}\}\simeq\mathbb{C}^{5*}\tag{2.8}$$

Recall also that L_1 consist of closed 2-forms with linear coefficients, that L_1 is an irreducible L_0-module and $\mathrm{L}_j=[\mathrm{L}_1[\cdots]]=\mathrm{L}_1^j$ for $j\geq1$. We take for the Borel subalgebra of $\mathrm{L}_0\simeq\mathrm{sl}_5$the subalgebra of the vector fields

$$\mathrm{Span}\{x_i\partial_j(1\leq i\leq j\leq5),\ x_i\partial_i-x_{i+1}\partial_{i+1}(i\in\overline{1,4})\}.\tag{2.9}$$

Given L_0 module V, we extend it to a L module by letting L_+ acts trivially, and define the induced module

$$M(V)=U(\mathrm{L})\otimes_{U(\mathrm{L}_0)}V\cong U(\mathrm{L}_-)V.\tag{2.10}$$

We denote by $\mathrm{V}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ the finite-dimensional irreducible L_0 -module with highest weight $\sum\limits_{i=1}^{4}\lambda_i\omega_i$, where ω_i $(i\in\overline{1,4})$ are the fundamental weights for sl_5. Let

$$M=M(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=M(V(\lambda_1,\lambda_2,\lambda_3,\lambda_4)) \tag{2.11}$$

denote the corresponding generalized Verma module over $\mathrm{E}(5,10)$.

Definition 2.1 *If $\xi\in M$ satisfies :*

$$(x_i\partial_{x_{i+1}}).\xi=0(i\in\overline{1,4}), \tag{2.12}$$

$$x_5d_{45}.\xi=0, \tag{2.13}$$

then we call ξ a singular vector for generalized verma module M of $E(5,10)$.

The aim of the following sections is to determine all the nontrivial singular vectors for $\mathrm{E}(5,10)$-module M.

3 Preliminary

In this section, we give some preparatory techniques about the irreducible tensor operators and the decomposition of tensor product module of simple Lie algebra.

Following the notations of Humphreys [H], let H be the Cartan subalgebra of simple Lie algebra L, and let $\Delta=\{\alpha_1,\cdots,\alpha_l\}$ be a base for the root system ϕ of H^*. The corresponding fundamental dominant weights $\{\omega_1,\cdots,\omega_l\}$ are defined from the root system via the form $<\cdot,\cdot>$ given by:

$$<\omega_i,\alpha_j>\equiv\frac{2(\omega_i,\alpha_j)}{(\alpha_j,\alpha_j)}=\delta_{ij}, \tag{3.1}$$

where $(\cdot,\cdot)$ denotes the inner product induced on H^* by the Killing form on H. Consider a basis $\{h_1,\cdots,h_l,x_\alpha,\alpha\in\phi\}$ of L where $h_1,\cdots,h_l$ is a basis for H and x_α is a nonzero element of the root space L_α. The dual basis may therefore be written $\{h^1,\cdots,h^l,x^\alpha,\alpha\in\phi\}$ where x^α is the unique element of $L_{-\alpha}$ which is dual to x_α under the Killing form of L. Write the universal Casimir element in the form:

$$c_L=\sum_{i=1}^{l}h_ih^i+\sum_{\alpha\in\phi}x_\alpha x^\alpha. \tag{3.2}$$

Let $V(\mu)$ be an irreducible highest weight module over L and let π_μ be the representation afforded by $V(\mu)$. Choose an ordeded basis $\{e_1,\cdots,e_d\}$ of $V(\mu)$, let $\pi_\mu(x)$ denote the matrix representing $x\in L$ on $V(\mu)$ with respect to this basis.

Definition 3.1 *We call a collection of linear operators $\{T_i:V\to W\mid i\in\overline{1,d}\}$ an irreducible tensor operator of rank μ if these components transform according to the rule:*

$$[x,T_i]=\pi_W(x)T_i-T_i\pi_V(x)=\sum_{j=1}^{d}\pi_\mu(x)_{ji}T_j,\ \ x\in L, \tag{3.3}$$

where V,W are (possibly infinite dimensional) L-modules and π_V (resp. π_W) is the representation afforded by V (resp. W).

Then we can define the following intertwining operator between L-modules $V(\mu)\otimes V$ and W:

$$T : V(\mu)\otimes V \to W,\ T(e_i\otimes v) = T_i(v),\ i\in\overline{1,d},\ v\in V. \tag{3.4}$$

In other words, $T\in \mathrm{Hom}_L(V(\mu)\otimes V, W)$ is an element of the set of all operators from $V(\mu)\otimes V$ to W commuting with the action of L.

Remark 3.2 In Section 5, we find an irreducible tensor operator of rank $\omega_1+\omega_2$ for simple Lie algebra sl_5, which play the center role in our determining all the singular vectors.

In the following two Lemmas, we record some well-known facts concerning the decomposition of tensor modules:

Lemma 3.3 (1) (cf. [H]) The α- string through any weight ν of $V(\mu)$ is of length $<\nu,\alpha>$, for $\alpha\in\phi$.

(2) (cf. [EG]) Denote $\mu_1,\cdots,\mu_m$ the weights occurring in $V(\mu)$ with multiplicities $n_1,\cdots,n_m$ respectively. For each $i\in\overline{1,m}$, let $V_i(\mu)$ denote the space of weight vectors of weight μ_i. The decomposition of the tensor product module $V(\mu)\otimes V(\lambda)$ is written:

$$V(\mu)\otimes V(\lambda) = \sum_{i=1}^{m} m(\lambda+\mu_i : \mu\otimes\lambda)V(\lambda+\mu_i),\ \lambda+\mu_i\in\Lambda^+, \tag{3.5}$$

where the multiplicities are given by

$$m(\lambda+\mu_i : \mu\otimes\lambda) = dim V_{i,\lambda}(\mu), \tag{3.6}$$

$$V_{i,\lambda}(\mu) = \{v\in V_i(\mu)\mid e_j^{<\lambda+\delta,\alpha_j>}v = 0, j\in\overline{1,l}.\} \tag{3.7}$$

(3) (cf. [EG]) Assume $\{e_{i,j}\mid j\in\overline{1,m(\lambda+\mu_i:\mu\otimes\lambda)}\}$ is a basis for the space $V_{i,\lambda}(\mu)$ and v_λ is the maximal weight vector of $V(\lambda)$. A full set of independent maximal weight states of weight $\lambda+\mu_i$ is given by the vectors:

$$\{P_i(e_{i,j}\otimes v_\lambda),\ j\in\overline{1,m(\lambda+\mu_i:\mu\otimes\lambda)}\},$$

where

$$P_i = \prod_{\mu_i<\sigma\le\mu}\frac{\tilde{c_L}-\chi_{\sigma+\lambda}(\tilde{c_L})}{\chi_{\mu_i+\lambda}(\tilde{c_L})-\chi_{\sigma+\lambda}(\tilde{c_L})},$$

$$\chi_{\sigma+\lambda}(\tilde{c_L}) = \frac{(\sigma+\lambda,\sigma+\lambda+2\delta)-(\mu,\mu+2\delta)-(\lambda,\lambda+2\delta)}{2},$$

$$\tilde{c_L} = \sum_{i=1}^{l}\pi_\mu(h_i)\otimes\pi_\lambda(h^i)+\sum_{\alpha\in\phi}\pi_\mu(x_\alpha)\otimes\pi_\lambda(x^\alpha). \tag{3.8}$$

(4) (cf. [MS]) The tensor product module $V(\mu)\otimes V(\lambda)$ is a cyclic module which is cyclically generated by the vector $v^\mu\otimes v_\lambda$, where v^μ is the lowest weight vector for $V(\mu)$ and v_λ is the highest weight vector for $V(\lambda)$.

For $\overrightarrow{a}=(a_1,\cdots,a_{n-1})\in\mathbb{N}^{n-1}$ and $0<k\in\mathbb{N}$, we denote

$$\overrightarrow{a}^*=(a_{n-1},a_{n-2},\cdots,a_1), \tag{3.9}$$

$$\begin{aligned} I(\overrightarrow{a},k)=\{&(a_1+c_1-c_2,a_2+c_2-c_3,\cdots,a_{n-1}+c_{n-1}-c_n)\mid c_i\in\mathbb{N}\\ &\text{such that}\ \sum_{i=1}^{n}c_i=k \text{ and } c_{s+1}\le a_s \text{for } s\in\overline{1,n-1}\}. \end{aligned} \tag{3.10}$$

Set

$$\omega_{\overrightarrow{a}}=\sum_{i=1}^{n-1}a_i\omega_i,\quad \text{for } \overrightarrow{a}\in\mathbb{N}^{n-1}, \tag{3.11}$$

Lemma 3.4 (Pieri's formula cf. [FH]) (1) For any $\overrightarrow{a}\in\mathbb{N}^{n-1}$, the tensor product of sl_n-module $V(\omega_{\overrightarrow{a}})$ with $V(k\omega_1)$ decomposes into a direct sum:

$$V(\omega_{\overrightarrow{a}})\otimes V(k\omega_1)=\bigoplus_{\overrightarrow{b}\in I(\overrightarrow{a},k)}V(\omega_{\overrightarrow{b}}). \tag{3.12}$$

(2) For sl_n, we have $V(\omega_{\overrightarrow{a}})^*=V(\omega_{\overrightarrow{a}^*})$ and

$$V(\omega_{\overrightarrow{a}})\otimes V(k\omega_{n-1})=\bigoplus_{\overrightarrow{b}\in I(\overrightarrow{a}^*,k)}V(\omega_{\overrightarrow{b}^*}). \tag{3.13}$$

In the rest of this section, we will concentrate on some special wedge and tensor modules for sl_5. Take $\{h_i=E_{i,i}-E_{i+1,i+1}(i\in\overline{1,4}),E_{ij}(1\le i\ne j\le 5)\}$ as a basis for Lie algebra sl_5. Then $\{\frac{h_i^*}{10}(i\in\overline{1,4}),\frac{E_{ji}}{10}(1\le i\ne j\le 5)\}$ is its dual basis via the Killing form, where

$$\begin{aligned} h_1^*&=\frac{4}{5}h_1+\frac{3}{5}h_2+\frac{2}{5}h_3+\frac{1}{5}h_4,\ h_2^*=\frac{3}{5}h_1+\frac{6}{5}h_2+\frac{4}{5}h_3+\frac{2}{5}h_4,\\ h_3^*&=\frac{2}{5}h_1+\frac{4}{5}h_2+\frac{6}{5}h_3+\frac{3}{5}h_4,\ h_4^*=\frac{1}{5}h_1+\frac{2}{5}h_2+\frac{3}{5}h_3+\frac{4}{5}h_4. \end{aligned} \tag{3.14}$$

And the Casimir operator c of the universal enveloping algebra of sl_5 is

$$c=\frac{1}{10}(\sum_{i=1}^{4}h_ih_i^*+\sum_{i\ne j\in\overline{1,5}}E_{i,j}.E_{j,i}). \tag{3.15}$$

Relative to the ordered basis $\omega_1,\omega_2,\omega_3,\omega_4$, the coordinates of the simple roots $\alpha_i(i\in\overline{1,4})$ are:

$$\alpha_1=(2,-1,0,0),\ \alpha_2=(-1,2,-1,0),\ \alpha_3=(0,-1,2,-1),\ \alpha_4=(0,0,-1,2). \tag{3.16}$$

And the killing form for the simple root $\alpha_i(i\in\overline{1,4})$ are:

$$(\alpha_i,\alpha_j)=\begin{cases}0, & |i-j|>1,\\ \frac{1}{5}, & i=j,\\ -\frac{1}{10}, & |i-j|=1.\end{cases} \tag{3.17}$$

Lemma 3.5 Assume $L=sl_5$, $C_L=c$ and $\sigma=\mu-\sum_{i=1}^{4}k_i\alpha_i$ in Lemma 3.4. Then $\chi_{\sigma+\lambda}(\tilde{c})$ in (3.6) is explicitly given by:

$$\begin{aligned} \chi_{\sigma+\lambda}(\tilde{c})=&\frac{\lambda_1(4\mu_1+3\mu_2+2\mu_3+\mu_4)}{50}+\frac{\lambda_2(3\mu_1+6\mu_2+4\mu_3+2\mu_4)}{50}+\frac{\lambda_3(2\mu_1+4\mu_2+6\mu_3+3\mu_4)}{50}\\ &+\frac{\lambda_4(\mu_1+2\mu_2+3\mu_3+4\mu_4)}{50}+\frac{\sum_{i=1}^{4}k_i^2-k_1k_2-k_2k_3-k_3k_4-\sum_{i=1}^{4}k_i-k_i(\lambda_i+\mu_i)}{10}. \end{aligned} \tag{3.18}$$

From (2.7) and (2.9), we know that $L_0 \simeq sl_5$. And L_0-module L_{-1} is isomorphic to fundamental module $V(\omega_2) = W$. The set of its weights and the basis for the corresponding weight space are tabulated in Table 1. The L_0-module L_1 is isomorphic to highest weight module $V(\omega_1 + \omega_2)$ with lowest weight vector $x_5 d_{45}$ (cf. Table 9).

Lemma 3.6 *The wedge module $\Lambda^k W$ ($k \in \overline{1,10}$) for sl_5 are decomposed multiplicity freely into irreducible components, which are listed in Table 2.*

Proof By Weyl's dimension formula, we get: $\dim V(\omega_1 + \omega_3) = 45$, $\dim V(2\omega_3) = 50$, $\dim V(2\omega_1 + \omega_4) = 70$, $\dim V(3\omega_1) = 35$, $\dim V(\omega_1 + \omega_3 + \omega_4) = 175$, $\dim V(2\omega_1 + \omega_3) = 126$, $\dim V(\omega_2 + 2\omega_4) = 126$. Since $\dim W = 10$, $\dim \wedge^k W = C_{10}^k$. Thus the decomposition follows through comparing the dimensions of both sides. $\square$

Lemma 3.7 *The tensor module $V(k\omega_4) \otimes \Lambda^n W$ ($k \in \mathbb{N}$, $n \in \overline{1,10}$) for sl_5 are decomposed into irreducible components, which are listed in Table 3.*

For any highest weight module $V(\mu)$ of simple Lie algebra sl_5, denote the set of its weights by $\Pi(\mu)$, which are listed by $\{\overrightarrow{w}_j^{\mu} \mid j \in \overline{1, |\Pi(\mu)|}\}$. Let $\{v_{j,k}^{\mu} \mid j \in \overline{1, |\Pi(\mu)|}, k \in \overline{1, \mathrm{mult}(\overrightarrow{w}_j^{\mu})}\}$ be the Verma basis for the weight space of weight $\overrightarrow{w}_j^{\mu}$, where $\mathrm{mult}(\overrightarrow{w}_j^{\mu})$ denotes the multiplicity of the weight $\overrightarrow{w}_j^{\mu}$.

Lemma 3.8 *For $\mu \in \{\omega_1+\omega_3, 2\omega_1+\omega_4, 3\omega_1\}$, the set $\Pi(\mu)$ of weights for $V(\mu)$ and their corresponding Verma bases for every weight space are listed in Table 5-Table 8 in the Appendix.*

Proof Assume $\mu = \sum_{i=1}^{4} m_i \omega_i$. The set $\Pi(\mu)$ is obtained by the algorithm from [W]. The Verma bases for the weight space with weight $\mu - \sum_{i=1}^{4} k_i \alpha_i$ are (cf. [LMNP], [RS]):

$$(f_1^{a_{10}} f_2^{a_9} f_3^{a_8} f_4^{a_7})(f_1^{a_6} f_2^{a_5} f_3^{a_4})(f_1^{a_3} f_2^{a_2}) f_1^{a_1} v_\mu, \tag{3.19}$$

where

$$a_{10} + a_6 + a_3 + a_1 = k_1, a_9 + a_5 + a_2 = k_2, a_8 + a_4 = k_3, a_7 = k_4,$$

$$0 \le a_1 \le m_1, 0 \le a_2 \le m_2 + a_1, 0 \le a_3 \le \min(m_2, a_2),$$

$$0 \le a_4 \le m_3 + a_2, 0 \le a_5 \le \min(m_3 + a_3, a_4), 0 \le a_6 \le \min(m_3, a_5),$$

$$0 \le a_7 \le m_4 + a_4, 0 \le a_8 \le \min(m_4 + a_5, a_7), 0 \le a_9 \le \min(a_4 + a_6, a_8), 0 \le a_{10} \le \min(m_4, a_9). \tag{3.20}$$

$\square$

Remark 3.9 The coordinates of the weights appearing in Table1, Table5-Table9 are with respect to the ordered basis $\omega_1, \omega_2, \omega_3, \omega_4$. The basis of every weight space appearing in these tables are Verma basis.

4 Singular vectors for GVM of $\mathbf{E}(5, 10)$

In Section 4.1, we analyze the detailed structure of the generalized Verma module M over $\mathrm{E}(5, 10)$. It turns out that there is a grading on M and each graded subspace is a finite dimensional sl_5- module

(cf. Equation (4.6) and (4.7)). Moreover, any singular vector for M is controlled by its leading term through an exponential-like differential operator (cf. Equation (4.27)). In section 4.2, we inductively prove that any leading term must satisfy three equations, i.e. (4.28), (4.35) and (4.37). Based on the Lemmas in Section 3, we simplify these three differential equations and prove that any singular vector is of degree less than or equal to four. Also, the leading term of any singular vector must lie in one of the tensor decomposition of four tensor product module for sl_5 (cf. Theorem 4.6).

4.1 Gradation for GVM

Set

$$T=\{0,1\},\ \ T'=\{(45),(35),(25),(15),(34),(24),(14),(23),(13),(12)\}. \tag{4.1}$$

Define order " $\prec$" on the set T' by:

$$(45)\prec(35)\prec(25)\prec(15)\prec(34)\prec(24)\prec(14)\prec(23)\prec(13)\prec(12). \tag{4.2}$$

For $\underline{n}=(n_{12},n_{13},n_{14},n_{24},n_{34},n_{15},n_{25},n_{35},n_{45})\in T^{10}$ and $\underline{m}\in\mathbb{N}^5$, we take the following notations:

$$\underline{n}\pm\varepsilon_{ij}=(n_{12},\cdots,n_{ij}\pm 1,\cdots,n_{45}),\ \underline{m}\pm\varepsilon_i=(m_1,\cdots,m_i\pm 1,\cdots,m_5). \tag{4.3}$$

Let

$$d^{\underline{n}}=d_{12}^{n_{12}}d_{13}^{n_{13}}d_{23}^{n_{23}}d_{14}^{n_{14}}d_{24}^{n_{24}}d_{34}^{n_{34}}d_{15}^{n_{15}}d_{25}^{n_{25}}d_{35}^{n_{35}}d_{45}^{n_{45}}, \tag{4.4}$$

$$\partial^{\underline{m}}=\partial_1^{m_1}\partial_2^{m_2}\partial_3^{m_3}\partial_4^{m_4}\partial_5^{m_5}. \tag{4.5}$$

Then the induced module M is spanned by $\{\partial^{\underline{m}}d^{\underline{n}}v_\nu\mid \underline{n}\in T^{10},\underline{m}\in\mathbb{N}^5,\nu\in\Pi(\lambda)\}$. Define

$$\partial^m\wedge^n V=\mathrm{Span}\{\partial^{\underline{m}}d^{\underline{n}}v\mid |\underline{m}|=m,|\underline{n}|=n\},\ M_k=\mathrm{Span}\{\partial^{\underline{m}}d^{\underline{n}}v\mid 2m+n=k\}. \tag{4.6}$$

Then

$$M=\bigoplus_{k\in\mathbb{N}}M_k. \tag{4.7}$$

Definition 4.1 We say any nonzero vector of M_k is of degree k.

The equations

$$[x_i\partial_{x_j},d_{kl}]=\delta_{j,k}d_{il}-\delta_{jl}d_{ik},[x_5d_{45},d_{12}]=x_5\partial_{x_3},\ [x_5d_{45},d_{13}]=-x_5\partial_{x_2},\ [x_5d_{45},d_{23}]=x_5\partial_{x_1},$$

$$[x_5d_{45},d_{i4}]=0(i\in\overline{1,3}),\ [x_5d_{45},d_{i5}]=0(i\in\overline{1,4}). \tag{4.8}$$

yield

$$L_0.\partial^m\wedge^n V\subseteq\partial^m\wedge^n V+\partial^{m+1}\wedge^{n-2}V,\ x_5d_{45}.M_k\subseteq M_{k-1}. \tag{4.9}$$

That is to say, every graded vector subspace M_k is an sl_5-module and every singular vector for E(5,10)-module M is in a certain graded subspace M_k.

In the following of this section, we consider the maximal vectors for sl_5-module M_k. On any linear vector space $\partial^m \wedge^n V$, we define the following linear operators:

$$(-1)^{|ij|}: \partial^m \wedge^n V \to \partial^m \wedge^n V; \partial^{\underline{m}} d^{\underline{n}} v \mapsto (-1)^{\sum\limits_{(kl)\prec(ij)} n_{kl}} \partial^{\underline{m}} d^{\underline{n}} v,$$

$$(-1)^{|ij,kl|}: \partial^m \wedge^n V \to \partial^m \wedge^n V; \partial^{\underline{m}} d^{\underline{n}} v \mapsto (-1)^{\sum\limits_{(kl)\prec(pq)\prec(ij)} n_{pq}} \partial^{\underline{m}} d^{\underline{n}} v,$$

$$y_{ij}\partial_{y_{kl}}: \partial^m \wedge^n V \to \partial^m \wedge^n V; \partial^{\underline{m}} d^{\underline{n}} v \mapsto n_{kl} \partial^{\underline{m}} d^{\underline{n}+\varepsilon_{ij}-\varepsilon_{kl}} v,$$

$$z_i: \partial^m \wedge^n V \to \partial^{m+1} \wedge^n V; \partial^{\underline{m}} d^{\underline{n}} v \mapsto \partial^{\underline{m}+\varepsilon_i} d^{\underline{n}} v,$$

$$\partial_{z_i}: \partial^m \wedge^n V \to \partial^{m-1} \wedge^n V; \partial^{\underline{m}} d^{\underline{n}} v \mapsto m_i \partial^{\underline{m}-\varepsilon_i} d^{\underline{n}} v,$$

$$E_{i,j}: \partial^m \wedge^n V \to \partial^m \wedge^n V; \partial^{\underline{m}} d^{\underline{n}} v \mapsto \partial^{\underline{m}} d^{\underline{n}} (E_{i,j}.v). \tag{4.10}$$

Set

$$(x_i\partial_{x_j})_0^{'} = \sum_{k\in\overline{1,5}, k\neq i,j} (-1)^{|ki,kj|} y_{ki}\partial_{y_{kj}}, \ (x_i\partial_{x_j})_0 = -z_j\partial_{z_i} + (x_i\partial_{x_j})_0^{'} + E_{i,j} (i\neq j); \tag{4.11}$$

$$(x_3\partial_{x_4})_{-2} = z_5 \partial_{y_{14}}\partial_{y_{24}}, \tag{4.12}$$

$$(x_4\partial_{x_5})_{-2} = z_1 \partial_{y_{25}}\partial_{y_{35}} + (-1)^{1+|15,35|} z_2 \partial_{y_{15}}\partial_{y_{35}} + z_3 \partial_{y_{15}}\partial_{y_{25}}. \tag{4.13}$$

Using these settings, we could formulate the equation (2.12) in the following explicit form:

$$x_1\partial_{x_2} = (x_1\partial_{x_2})_0, \quad x_2\partial_{x_3} = (x_2\partial_{x_3})_0,$$

$$x_3\partial_{x_4} = (x_3\partial_{x_4})_0 + (x_3\partial_{x_4})_{-2}, \ x_4\partial_{x_5} = (x_4\partial_{x_5})_0 + (x_4\partial_{x_5})_{-2}. \tag{4.14}$$

According to the Cartan subalgebra of L_0, M can be decomposed to the following direct sum of subspaces:

$$M = \bigoplus_{\mu\in\Gamma} M^\mu, \ M^\mu = \mathrm{Span}\{\partial^{\underline{m}} d^{\underline{n}} v_\nu \mid (x_i\partial_{x_i} - x_{i+1}\partial_{x_{i+1}}).\partial^{\underline{m}} d^{\underline{n}} v_\nu = \mu_i \partial^{\underline{m}} d^{\underline{n}} v_\nu\}, \tag{4.15}$$

where

$$\mu_i = m_{i+1} - m_i + t_i(\underline{n}) + \nu_i,$$

$$t_1(\underline{n}) = n_{13}+n_{14}+n_{15}-n_{23}-n_{24}-n_{25}, \ t_2(\underline{n}) = n_{12}+n_{24}+n_{25}-n_{13}-n_{34}-n_{35},$$

$$t_3(\underline{n}) = n_{13}+n_{23}+n_{35}-n_{14}-n_{24}-n_{45}, \ t_4(\underline{n}) = n_{14}+n_{24}+n_{34}-n_{15}-n_{25}-n_{35}. \tag{4.16}$$

For any vectors $v \in M^\mu$, we say that it is of weight μ and denote $\mathrm{wt}(v) = \mu$, $|\mathrm{wt}(v)| = |\mu| = \sum\limits_{i=1}^{4} \mu_i$.

Proposition 4.2 *The differential operators* $(x_i\partial_{x_j})_0 (1 \le i \neq j \le 5)$ *and* $x_i\partial_{x_i} - x_{i+1}\partial_{x_{i+1}} (i \in \overline{1,4})$ *give every vector space* $\partial^m \wedge^n V$ *an* sl_5*-module structure, which is isomorphic to tensor module* $V(m\omega_4) \otimes \wedge^n W \otimes V$ *for* sl_5.

Proof The module isomorphism is given by:

$$\phi: V(m\omega_4) \otimes \wedge^n W \otimes V \to \partial^m \wedge^n V; \partial^{\underline{m}} \otimes (d_{i_1j_1} \wedge \cdots \wedge d_{i_nj_n}) \otimes v \mapsto \partial^{\underline{m}} d_{i_1j_1} \cdots d_{i_nj_n} v. \tag{4.17}$$

$\square$

Denote

$$\Gamma_k = \{(m,n) \in \mathbb{N}^2 \mid 2m+n=k\} \tag{4.18}$$

For any $(m,n) \in \Gamma_k$, let

$$\Gamma_k^{(m,n)} = \{(m',n') \in \Gamma_k \mid m' \geq m\} \tag{4.19}$$

Assume $\xi \in M_k$ is any $E(5,10)$ singular vector. Then there exists $(m,n) \in \Gamma_k$ such that

$$\xi \in \bigoplus_{(m'n')\in\Gamma_k^{(m,n)}} \partial^{m'} \wedge^{n'} V \tag{4.20}$$

For emphasis, we write

$$\xi = \xi^{m,n} = \sum_{(m'n')\in\Gamma_k^{(m,n)}} \xi_{m',n'} \tag{4.21}$$

We say that $\xi_{m,n}$ is *the leading term* of $\xi^{m,n}$. It follows from (4.14) that $\xi^{m,n}$ must satisfy the following equations inductively:

$$(x_i\partial_{x_{i+1}})_0.\xi_{m,n} = 0 (i \in \overline{1,4}),$$

$$(x_i\partial_{x_{i+1}})_{-2}.\xi_{m',n'} + (x_i\partial_{x_{i+1}})_0.\xi_{m'+1,n'-2} = 0,\ i \in \overline{1,4},\ (m',n') \in \Gamma_k^{(m,n)}. \tag{4.22}$$

Remark 4.3 *From Proposition 4.2 and (4.22), we derive that the leading term $\xi_{m,n}$ of any singular vector $\xi = \xi^{m,n}$ is also a singular vector of the tensor product module $V(m\omega_4) \otimes \wedge^n V(\omega_2) \otimes V(\lambda)$ for simple Lie algebra sl_5. In the following, we will point out that any singular vector $\xi = \xi^{m,n}$ is completely controlled by its leading term $\xi_{m,n}$ through certain exponential-like differential operator.*

Set

$$P = \sum_{(kl)\prec(ij)\in T', m\in\overline{1,5}} \varepsilon_{mijkl}(-1)^{|ij,kl|} z_m \partial_{y_{ij}} \partial_{y_{kl}}, \tag{4.23}$$

where ε_{mijkl} is defined in (2.6). The operator P is checked to satisfy the following equations:

$$[(x_3\partial_{x_4})_{-2}, P] = 0, [(x_4\partial_{x_5})_{-2}, P] = 0,\ [(x_1\partial_{x_2})_0, P] = 0, [(x_2\partial_{x_3})_0, P] = 0,$$

$$[(x_3\partial_{x_4})_0, P] = 2(x_3\partial_{x_4})_{-2},\ [(x_4\partial_{x_5})_0, P] = 2(x_4\partial_{x_5})_{-2}. \tag{4.24}$$

Inductively,

$$[(x_i\partial_{x_{i+1}})_0, P^k] = [x_i\partial_{x_{i+1}}, P^k] = 2kP^{k-1}(x_i\partial_{x_{i+1}})_{-2}, k \in \mathbb{N}. \tag{4.25}$$

It implies

$$x_i\partial_{x_{i+1}}.e^{-\frac{1}{2}P}\xi_{m,n} = 0, \quad (x_i\partial_{x_{i+1}})_0.e^{\frac{1}{2}P}\xi^{m,n} = 0. \tag{4.26}$$

Thus we prove the following formula:

Proposition 4.3 *Assume $\xi^{m,n} = \sum_{(m'n')\in\Gamma_k^{(m,n)}} \xi_{m',n'} \in M_k$ is any singular vector for $E(5,10)$-module M, then*

$$\xi^{m,n} = e^{-\frac{1}{2}P}\xi_{m,n}. \tag{4.27}$$

4.2 Singular vectors for GVM

In this section, we continue the discussion concerning the equation (2.13) in Definition 2.1. Recall the notations in (4.10), set

$$(x_5d_{45})_1 = (-1)^{1+|45|}\partial_{z_5}y_{45}, \tag{4.28}$$

$$\begin{aligned}(x_5d_{45})_{-1} \;\;=\;\; & -z_3\partial_{z_5}\partial_{y_{12}} + (-1)^{|13|}z_2\partial_{z_5}\partial_{y_{13}} + (-1)^{1+|23|}z_1\partial_{z_5}\partial_{y_{23}} \\ + \;\; & \partial_{y_{12}}E_{53} + (-1)^{1+|13|}\partial_{y_{13}}E_{52} + (-1)^{|23|}\partial_{y_{23}}E_{51} \\ + \;\; & (-1)^{|13,15|}y_{15}\partial_{y_{12}}\partial_{y_{13}} + (-1)^{|23,25|}y_{25}\partial_{y_{12}}\partial_{y_{23}} + (-1)^{|13|+|23,35|}y_{35}\partial_{y_{13}}\partial_{y_{23}} \\ + \;\; & (-1)^{1+|34,45|}y_{45}\partial_{y_{12}}\partial_{y_{34}} + (-1)^{1+|23|+|14,45|}y_{45}\partial_{y_{23}}\partial_{y_{14}} + (-1)^{|13|+|24,45|}y_{45}\partial_{y_{13}}\partial_{y_{24}},\end{aligned} \tag{4.29}$$

$$\begin{aligned}(x_5d_{45})_{-3} \;\;=\;\; & (-1)^{|23,34|}z_1\partial_{y_{12}}\partial_{y_{23}}\partial_{y_{34}} + (-1)^{1+|23,24|+|13|}z_1\partial_{y_{13}}\partial_{y_{23}}\partial_{y_{24}} \\ + \;\; & (-1)^{|13|}z_2\partial_{y_{13}}\partial_{y_{23}}\partial_{y_{14}} + (-1)^{1+|13,34|}z_2\partial_{y_{12}}\partial_{y_{13}}\partial_{y_{34}} \\ + \;\; & (-1)^{|13,24|}z_3\partial_{y_{12}}\partial_{y_{13}}\partial_{y_{24}} - z_3\partial_{y_{12}}\partial_{y_{23}}\partial_{y_{14}} - z_4\partial_{y_{12}}\partial_{y_{13}}\partial_{y_{23}}.\end{aligned} \tag{4.30}$$

It follows from the equation (4.8) that

$$x_5d_{45}.\partial^m \wedge^n V \subseteq \partial^{m-1}\wedge^{n+1} V + \partial^m \wedge^{n-1} V + \partial^{m+1}\wedge^{n-3} V,\; x_5d_{45}.M_k \subseteq M_{k-1},$$

$$x_5d_{45} = (x_5d_{45})_1 + (x_5d_{45})_{-1} + (x_5d_{45})_{-3}. \tag{4.31}$$

Furthermore, $\xi^{m,n}$ must satisfy the following equations inductively:

$$(x_5d_{45})_1.\xi_{m,n} = 0,\; (x_5d_{45})_{-1}.\xi_{m,n} + (x_5d_{45})_1.\xi_{m+1,n-2} = 0,$$

$$(x_5d_{45})_{-3}.\xi_{m,n} + (x_5d_{45})_{-1}.\xi_{m+1,n-2} + (x_5d_{45})_1.\xi_{m+2,n-4} = 0,$$

$$(x_5d_{45})_{-3}.\xi_{m',n'} + (x_5d_{45})_{-1}.\xi_{m'+1,n'-2} + (x_5d_{45})_1.\xi_{m'+2,n'-4} = 0, \quad \text{for any } (m',n') \in \Gamma_k^{(m,n)}. \tag{4.32}$$

Applying (4.27), the leading term $\xi_{m,n}$ should be killed by the following three operators:

$$(x_5d_{45})_1,\; (x_5d_{45})_{-1} + (x_5d_{45})_1(-\frac{1}{2}P),\; (x_5d_{45})_{-3} + (x_5d_{45})_{-1}(-\frac{1}{2}P) + (x_5d_{45})_1(\frac{1}{8}P^2). \tag{4.33}$$

We can reduce the last two differential operators to be of more explicit forms. Indeed, the following relations are easily checked:

$$\begin{aligned}[(x_5d_{45})_1, P] \;\;=\;\; & -z_3\partial_{z_5}\partial_{y_{12}} + (-1)^{|13|}z_2\partial_{z_5}\partial_{y_{13}} + (-1)^{1+|23|}z_1\partial_{z_5}\partial_{y_{23}} \\ + \;\; & (-1)^{1+|34,45|}y_{45}\partial_{y_{12}}\partial_{y_{34}} + (-1)^{1+|23|+|14,45|}y_{45}\partial_{y_{23}}\partial_{y_{14}} + (-1)^{|13|+|24,45|}y_{45}\partial_{y_{13}}\partial_{y_{24}},\end{aligned}$$

$$[[(x_5d_{45})_1, P], P] = 2(x_5d_{45})_{-3} + 2z_4\partial_{y_{12}}\partial_{y_{13}}\partial_{y_{23}},\; [(x_5d_{45})_{-1}, P] = 3(x_5d_{45})_{-3}. \tag{4.34}$$

Therefore,

$$
\begin{aligned}
&[(x_5d_{45})_{-1}+(x_5d_{45})_1(-\frac{1}{2}P)]\xi_{m,n}\\
&=\{(x_5d_{45})_{-1}-\frac{1}{2}P(x_5d_{45})_1-\frac{1}{2}[(x_5d_{45})_1,P]\}\xi_{m,n}\\
&\overset{\text{by}(4.34)}{=}[-\frac{1}{2}z_3\partial_{z_5}\partial_{y_{12}}+\frac{1}{2}(-1)^{|13|}z_2\partial_{z_5}\partial_{y_{13}}+(-1)^{1+|23|}\frac{1}{2}z_1\partial_{z_5}\partial_{y_{23}}\\
&+\partial_{y_{12}}E_{53}+(-1)^{1+|13|}\partial_{y_{13}}E_{52}+(-1)^{|23|}\partial_{y_{23}}E_{51}\\
&+(-1)^{|13,15|}y_{15}\partial_{y_{12}}\partial_{y_{13}}+(-1)^{|23,25|}y_{25}\partial_{y_{12}}\partial_{y_{23}}\\
&+(-1)^{|13|+|23,35|}y_{35}\partial_{y_{13}}\partial_{y_{23}}+\frac{1}{2}(-1)^{1+|34,45|}y_{45}\partial_{y_{12}}\partial_{y_{34}}\\
&+\frac{1}{2}(-1)^{1+|23|+|14,45|}y_{45}\partial_{y_{23}}\partial_{y_{14}}+\frac{1}{2}(-1)^{|13|+|24,45|}y_{45}\partial_{y_{13}}\partial_{y_{24}}]\xi_{m,n}.\\
&\overset{\text{by}(4.11)}{=}[(E_{53}+\frac{1}{2}(x_5\partial_{x_3})'_0-\frac{1}{2}z_3\partial_{z_5})\partial_{y_{12}}+(E_{52}+\frac{1}{2}(x_5\partial_{x_2})'_0-\frac{1}{2}z_2\partial_{z_5})(-1)^{1+|13|}\partial_{y_{13}}\\
&+(E_{51}+\frac{1}{2}(x_5\partial_{x_1})'_0-\frac{1}{2}z_1\partial_{z_5})(-1)^{|23|}\partial_{y_{23}}]\xi_{m,n} \qquad (4.35)
\end{aligned}
$$

Hence,

$$P(x_5d_{45})_{-1}\xi_{m,n}=\frac{1}{2}P(x_5d_{45})_1P\xi_{m,n}=\frac{1}{2}P[(x_5d_{45})_1,P]\xi_{m,n}. \qquad (4.36)$$

Furthermore, (4.34) and (4.35) imply that

$$
\begin{aligned}
&[(x_5d_{45})_{-3}+(x_5d_{45})_{-1}(-\frac{1}{2}P)+(x_5d_{45})_1(\frac{1}{8}P^2)]\xi_{m,n}\\
&=\{(x_5d_{45})_{-3}-\frac{3}{2}(x_5d_{45})_{-3}-\frac{1}{2}P(x_5d_{45})_{-1}\\
&+\frac{1}{8}[[(x_5d_{45})_1,P],P]+\frac{1}{8}P[(x_5d_{45})_1,P]+\frac{1}{8}P(x_5d_{45})_1P\}\xi_{m,n}\\
&\overset{\text{by}(4.34),(4.35),(4.36)}{=}[-\frac{1}{4}(x_5d_{45})_{-3}+\frac{1}{4}z_4\partial_{y_{12}}\partial_{y_{13}}\partial_{y_{23}}]\xi_{m,n}\\
&=\frac{1}{4}[z_1(-1)^{|13|+|23,24|}\partial_{y_{13}}\partial_{y_{23}}\partial_{y_{24}}-z_1(-1)^{|23,34|}\partial_{y_{12}}\partial_{y_{23}}\partial_{y_{34}}\\
&+z_2(-1)^{|13,34|}\partial_{y_{12}}\partial_{y_{13}}\partial_{y_{34}}-z_2(-1)^{|13|}\partial_{y_{13}}\partial_{y_{23}}\partial_{y_{14}}\\
&+z_3\partial_{y_{12}}\partial_{y_{23}}\partial_{y_{14}}-z_3(-1)^{|13,24|}\partial_{y_{12}}\partial_{y_{13}}\partial_{y_{24}}+2z_4\partial_{y_{12}}\partial_{y_{13}}\partial_{y_{23}}]\xi_{m,n}. \qquad (4.37)
\end{aligned}
$$

Denote the set of all the highest weight vectors for tensor modules $V(m\omega_4)\bigotimes\wedge^nW$ by

$$S_{m,n}=\{e^1_{m,n},\cdots,e^{\nu(m,n)}_{m,n}\} \qquad (4.38)$$

By Lemma 3.3, any singular vector $\xi_{m,n}$ of the tensor product module $V(m\omega_4)\bigotimes\bigwedge^nV(\omega_2)\bigotimes V(\lambda)$ for sl_5 can be written by the following form:

$$\xi_{m,n}=e^i_{m,n}\otimes v_\vartheta+\cdots. \qquad (4.39)$$

We consider the set

$$
\begin{aligned}
S'_{m,n}=\{e^i_{m,n}\in S_{m,n}\mid &(-1)^{1+|45|}\partial_{z_5}y_{45}.\phi(e^i_{m,n}\otimes v)=0,\\
&[(x_5d_{45})_{-3}+(x_5d_{45})_{-1}(-\frac{1}{2}P)+(x_5d_{45})_1(\frac{1}{8}P^2)].\phi(e^i_{m,n}\otimes v)=0,\ \forall\ v\in V(\lambda)\}. \qquad (4.40)
\end{aligned}
$$

Proposition 4.4 *All the non empty set of $S'_{m,n}$ are listed in the following:*

$$S'_{0,0}=\{1\}, S'_{0,1}=\{d_{12}\}, S'_{0,2}=\{d_{12}\wedge d_{13}\}, S'_{0,3}=\{d_{12}\wedge d_{13}\wedge d_{14}\}, S'_{0,4}=\{d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{15}\}.$$

Proof Let $V(\mu)$ be any highest weight module appearing in the decomposition of the sl_5 wedge module $\wedge^n V(\omega_2)$ (cf. Table 2). And the highest weights appearing in the decomposition of $V(m\omega_4)\otimes V(\mu)$ are listed in Table 3. By Lemma 3.3, the maximal vector in the tensor module $V(m\omega_4)\otimes V(\mu)$ is written as:

$$\partial_5^m\otimes l_\mu+\sum_{\underline{q}\in\mathbb{N}^5}\partial^{\underline{q}}\otimes v_{\underline{q}}, \tag{4.41}$$

where l_μ satisfies $E_{12}l_\mu=0, E_{23}l_\mu=0, E_{34}l_\mu=0, E_{45}^{m+1}l_\mu=0$. By detailed calculation, we get all the l_μ, which are listed in Table-4. A straightforward but messy check case by case shows that the assertion holds. $\square$

Remark 4.5 *For the 10-tuple $d_{12}\wedge d_{13}\wedge\cdots\wedge d_{45}$, we use the notation $\hat{d}_{i_1j_1}\wedge\hat{d}_{i_2j_2}\wedge\cdots\wedge\hat{d}_{i_kj_k}$ to denote the (10-k)-tuple where $d_{i_1j_1},\cdots,d_{i_kj_k}$ have been omitted in Table 4.*

To summarize Proposition 4.3 and Proposition 4.4, we have proved the following statement in this section:

Theorem 4.6 *Any singular vector for $E(5,10)$-module M is of the form:*

$$\xi^{0,n}=e^{-\frac{1}{2}P}\xi_{0,n},\quad n\in\overline{1,4}$$

where the leading term $\xi_{0,n}$ satisfies the equation:

$$[(E_{53}+\frac{1}{2}(x_5\partial_{x_3})'_0)\partial_{y_{12}}+(E_{52}+\frac{1}{2}(x_5\partial_{x_2})'_0)(-1)^{1+|13|}\partial_{y_{13}}+(E_{51}+\frac{1}{2}(x_5\partial_{x_1})'_0)(-1)^{|23|}\partial_{y_{23}}].\xi_{0,n}=0. \tag{4.42}$$

Moreover, $\xi_{0,n}$ is the maximal vector lying in one of the following sl_5-tensor modules:

$$\xi_{0,1}\in V(\omega_2)\otimes V(\lambda), \xi_{0,2}\in V(\omega_1+\omega_3)\otimes V(\lambda), \xi_{0,3}\in V(2\omega_1+\omega_4)\otimes V(\lambda), \xi_{0,4}\in V(3\omega_1)\otimes V(\lambda). \tag{4.43}$$

5 Singular vectors degree by degree

In this section, we work out all the singular vectors in Theorem 4.6 explicitly degree by degree. Before turning to the calculation, we introduce some formula which we are going to use in the remainder of this section.

Recall that we could endow any vector space $\Lambda^m V$ an sl_5- module structure with the action $(x_i\partial_{x_i})_0'-(x_{i+1}\partial_{x_{i+1}})_0'$ $(i\in\overline{1,4})$, $(x_i\partial_{x_j})_0'$ $(i\neq j)$, which is isomorphic to the tensor product module $\wedge^m V(\omega_2)\otimes V(\lambda)$ in Section 4.1. Now we define the following differential operator on the sl_5- module $\Lambda^m V$:

$$\tilde{c}=\frac{1}{10}[\sum_{i=1}^{4}((x_i\partial_{x_i})_0'-(x_{i+1}\partial_{x_{i+1}})_0')h_i^*+\sum_{1\le i\neq j\le 5}(x_i\partial_{x_j})_0'E_{ji}], \tag{5.1}$$

$$T_{i,jkl}=[E_{ij}+\frac{1}{2}(x_i\partial_{x_j})'_0](-1)^{|kl|}\partial_{y_{kl}}+[E_{ik}+\frac{1}{2}(x_i\partial_{x_k})'_0](-1)^{1+|jl|}\partial_{y_{jl}}+[E_{il}+\frac{1}{2}(x_i\partial_{x_l})'_0](-1)^{|jk|}\partial_{y_{jk}}. \quad (5.2)$$

Lemma 5.1 Assume $Q^0_{ij}\in \mathrm{Span}_{\mathbb{F}}\{y_{ij}\partial_{y_{kl}}\mid 1\leq i<j\leq 5,\ 1\leq k<l\leq 5\}$. Then

$$\sum_{1\leq i<j\leq 5} Q^0_{ij}.(-1)^{|ij|}\partial_{y_{ij}}.\tilde{c}|_{\Lambda^m V}=\sum_{1\leq i<j\leq 5} Q^1_{ij}(-1)^{|ij|}\partial_{y_{ij}}|_{\wedge^m V},$$

where

$$Q^1_{ij}=Q^0_{ij}(\tilde{c}+\sum_{k=1}^4\frac{s^{ij}_k}{10}h^*_k)+\frac{1}{10}\sum_{m\neq i,j}(Q^0_{im}E_{jm}-Q^0_{jm}E_{im}),$$

$$s^{ij}_k \overset{\text{by}(4.16)}{=} t_k(\underline{n})-t_k(\underline{n}-\epsilon_{ij}) \quad (5.3)$$

for any $k\in\overline{1,4},\underline{n}\in T^{10},(ij)\in S'$.

Proof Indeed, the formula (5.3) follows from:

$$[(-1)^{|ij|}\partial_{y_{ij}},\tilde{c}]|_{\wedge^m V}=\frac{1}{10}(\sum_{k\neq i,j}(-1)^{|ik|}\partial_{y_{ik}}E_{kj}-\sum_{k\neq i,j}(-1)^{|jk|}\partial_{y_{jk}}E_{ki}+\sum_{k=1}^4 s^{ij}_k(-1)^{|ij|}\partial_{y_{ij}}h^*_k)|_{\wedge^m V}. \quad (5.4)$$

□

Lemma 5.2 We could define the following intertwining operators between the sl_5-module $V(\omega_1+\omega_2)\otimes\Lambda^m V$ and $\Lambda^{m-1}V$ by:

$$T^m: V(\omega_1+\omega_2)\otimes\Lambda^m V\to\Lambda^{m-1}V; v^{\omega_1+\omega_2}_{30,1}\otimes\xi\mapsto T_{5,123}(\xi), \quad (5.5)$$

where $v^{\omega_1+\omega_2}_{30,1}$ is the lowest weight vector for $V(\omega_1+\omega_2)$ (cf. Table 9) and ξ is any maximal vector in sl_5-module $\Lambda^m V$.

Proof Since the sl_5-module $V(\omega_1+\omega_2)\otimes\Lambda^m V$ is generated by such vectors of $v^{\omega_1+\omega_2}_{30,1}\otimes\xi$ by part (4) of Lemma 3.3, the assertion follows from the following formula:

$$[(x_{i+1}\partial_{x_i})'_0,T_{5,123}]|_{\wedge^m V}=0,\ i\in\overline{1,4},$$

$$[(x_s\partial_{x_t})'_0,T_{i,jkl}]|_{\wedge^m V}=\delta_{t,i}T_{s,jkl}-\delta_{s,j}T_{i,tkl}-\delta_{s,k}T_{i,jtl}-\delta_{s,l}T_{i,jkt}. \quad (5.6)$$

□

5.1 Singular vectors of degree one

Theorem 5.3 All the possible degree one singular vectors are listed in the following:

$$d_{12}v_\lambda,\ \text{where } \lambda=(m,n,0,0),\ m,n\in\mathbb{N};$$

$$\prod_{\mathrm{wt}(d_{15})<\sigma\leq\omega_2}\frac{\tilde{c}-\chi_{\sigma+\lambda}(\tilde{c})}{\chi_{\mathrm{wt}(d_{15})+\lambda}(\tilde{c})-\chi_{\sigma+\lambda}(\tilde{c})}.d_{15}v_\lambda,\ \text{where } \lambda=(m,0,0,n),\ m\in\mathbb{N},\ 1\leq n\in\mathbb{N};$$

$$\prod_{\mathrm{wt}(d_{45})<\sigma\leq\omega_2}\frac{\tilde{c}-\chi_{\sigma+\lambda}(\tilde{c})}{\chi_{\mathrm{wt}(d_{45})+\lambda}(\tilde{c})-\chi_{\sigma+\lambda}(\tilde{c})}.d_{45}v_\lambda,\ \text{where } \lambda=(0,0,m,n),\ 1\leq m\in\mathbb{N},\ n\in\mathbb{N}.$$

Proof The leading term of any singular vector of degree one can be written as

$$\xi_{0,1}=\sum_{1\leq i<j\leq 5} d_{ij}v_{ij},\ v_{ij}\in V(\lambda), \tag{5.7}$$

which should satisfy :

$$T_{5,123}.\xi_{0,1}=[\partial_{y_{12}}E_{53}+(-1)^{1+|13|}\partial_{y_{13}}E_{52}+(-1)^{|23|}\partial_{y_{23}}E_{51}].\xi_{0,1}=0, \tag{5.8}$$

i.e.

$$E_{53}v_{12}-E_{52}v_{13}+E_{51}v_{23}=0. \tag{5.9}$$

Note that $(x_i\partial_{x_j})_0.\xi_{0,1}=0$ $(1\leq i<j\leq 5)$ imply that

$$v_{13}=-E_{23}v_{12},\ v_{23}=E_{13}v_{12}=-E_{12}v_{13},\ v_{14}=-E_{34}v_{13},\ v_{15}=-E_{25}v_{12}=-E_{35}v_{13},$$

$$v_{25}=E_{15}v_{12}=-E_{35}v_{23},\ v_{35}=E_{15}v_{13}=-E_{23}v_{25}=E_{25}v_{23},\ v_{45}=-E_{34}v_{35}. \tag{5.10}$$

Obviously, $v_{12}\neq 0$.

Case 1. $\mathrm{wt}(\xi_{0,1})=\mathrm{wt}(d_{12}v_\lambda)$.

In this case, $v_{13}=v_{23}=0$, $v_{12}=v_\lambda$. And (5.9) implies that $E_{5,3}.v_{12}=E_{5,3}.v_\lambda=0$. That is to say, $\lambda=(m,n,0,0),(m,n)\in\mathbb{N}^2$.

Case 2. $\mathrm{wt}(\xi_{0,1})\in\{\mathrm{wt}(d_{13}v_\lambda),\mathrm{wt}(d_{14}v_\lambda),\mathrm{wt}(d_{15}v_\lambda)\}$.

In these three cases, we have $v_{23}=0, v_{13}\neq 0$.

Case 2.1 $wt(\xi_{0,1})\in\{\mathrm{wt}(d_{13}v_\lambda),\mathrm{wt}(d_{14}v_\lambda)\}$

In these two cases, $v_{15}=0, v_{13}\neq 0$, $(h_2+h_3+h_4).v_{13}=(\lambda_2+\lambda_3+\lambda_4)v_{13}$. Hence,

$$\begin{aligned}0&=E_{25}(E_{53}.v_{12}-E_{52}.v_{13})=(E_{23}+E_{53}E_{25})v_{12}-(h_2+h_3+h_4+E_{52}E_{25})v_{13}\\&=-(1+h_2+h_3+h_4)v_{13}-E_{53}v_{15}=-(1+h_2+h_3+h_4)v_{13}=-(1+\lambda_2+\lambda_3+\lambda_4)v_{13}\end{aligned} \tag{5.11}$$

provides a contradiction.

Case 2.2 $wt(\xi_{0,1})=\mathrm{wt}(d_{15}v_\lambda)$

In this case, $v_{15}=v_\lambda$ and $\mathrm{wt}(v_{13})=\lambda-\alpha_3-\alpha_4$. And

$$0=E_{25}(E_{53}.v_{12}-E_{52}.v_{13})=-(\lambda_2+\lambda_3+\lambda_4)v_{13}-E_{53}v_{15},$$

$$0=E_{35}E_{25}(E_{53}.v_{12}-E_{52}.v_{13})=-E_{35}(\lambda_2+\lambda_3+\lambda_4)v_{13}-E_{35}E_{53}v_{15}=(\lambda_2+\lambda_3+\lambda_4-h_3-h_4)v_{15}=\lambda_2v_{15} \tag{5.12}$$

forces $\lambda_2=0$.

Case 2.2.1. $\lambda_3=0,\ \lambda_4>0$.

Suppose

$$T_{5,123}.\tilde{c}^k=\sum_{1\leq i<j\leq 5} Q_{ij}^{k-1}(-1)^{|ij|}\partial_{y_{ij}}. \tag{5.13}$$

By (5.3), $Q^1_{15}v_\lambda = (Q^0_{12}E_{52} + Q^0_{13}E_{53})v_\lambda = 0$. Then

$$T_{5,123}.\xi_{0,1} = [\partial_{y_{12}}E_{53} + (-1)^{1+|13|}\partial_{y_{13}}E_{52} + (-1)^{|23|}\partial_{y_{23}}E_{51}].\tilde{c}.d_{15}v_\lambda = Q^1_{15}.v_\lambda = 0. \tag{5.14}$$

Therefore, $\lambda = (m, 0, 0, n), m \in \mathbb{N}, n > 0$.

Case 2.2.2. $\lambda_3 > 0,\ \lambda_4 > 0$.

Note that $E_{54}v_\lambda \neq 0$. Since $T_{5,123}.\tilde{c}.d_{15}v_\lambda = Q^1_{15}.v_\lambda = 0$, the equation (3.8) implies

$$T_{5,123}.\xi_{0,1} = T_{5,123}.\tilde{c}^2.d_{15}v_\lambda = Q^2_{15}.v_\lambda = (E_{53}E_{42}E_{54} - E_{52}E_{43}E_{54})v_\lambda = 0; \tag{5.15}$$

which yields

$$0 = E_{34}E_{25}(E_{53}E_{42}E_{54} - E_{52}E_{43}E_{54})v_\lambda = -\lambda_3(1 + \lambda_3 + \lambda_4)E_{54}v_\lambda. \tag{5.16}$$

A contradiction arises.

Case 3. $\mathrm{wt}(\xi_{0,1}) \in \{\mathrm{wt}(d_{23}v_\lambda), \mathrm{wt}(d_{24}v_\lambda), \mathrm{wt}(d_{34}v_\lambda), \mathrm{wt}(d_{25}v_\lambda), \mathrm{wt}(d_{35}v_\lambda), \mathrm{wt}(d_{45}v_\lambda)\}$.

In these cases, $v_{13} \neq 0, v_{23} \neq 0$. Set

$$Q = E_{53}.v_{12} - E_{52}.v_{13} + E_{51}.v_{23},\ E_{15}Q = Q_1,\ E_{25}Q_1 = Q_2, E_{35}Q_1 = Q_2'\ . \tag{5.17}$$

Then

$$0 = Q_1 = E_{13}v_{12} + E_{53}E_{15}v_{12} - E_{12}v_{13} - E_{52}E_{15}v_{13} + (h_1 + h_2 + h_3 + h_4)v_{23}. \tag{5.18}$$

Case 3.1. $wt(\xi_{0,1}) \in \{\mathrm{wt}(d_{23}v_\lambda), \mathrm{wt}(d_{24}v_\lambda), \mathrm{wt}(d_{34}v_\lambda)\}$.

In these three cases, we have $v_{23} \neq 0$, $v_{25} = v_{35} = 0$. So

$$0 = Q_1 = (2 + |\mathrm{wt}(v_{23})|)v_{23} \tag{5.19}$$

induces a contradiction.

Case 3.2. $wt(\xi_{0,1}) = \mathrm{wt}(d_{25}v_\lambda)$.

In this case, $v_{35} = 0$ and $\mathrm{wt}(v_{23}) = \lambda - \alpha_3 - \alpha_4$. So

$$0 = Q_1 = (2 + |\mathrm{wt}(v_{23})|)v_{23} + E_{53}v_{25}, 0 = Q_2' = -(|\lambda| + 2 - h_3 - h_4)v_{25} = -(\lambda_1 + \lambda_2 + 1)v_{25} \tag{5.20}$$

force $\lambda_1 + \lambda_2 + 1 = 0$. A contradiction arises.

Case 3.3. $wt(\xi_{0,1}) \in \{\mathrm{wt}(d_{35}v_\lambda), \mathrm{wt}(d_{45}v_\lambda)\}$.

In these two cases, $v_{35} \neq 0$. And the equations

$$0 = Q_1 = (2+|\mathrm{wt}(v_{23})|)v_{23} + E_{53}v_{25} - E_{52}v_{35}, 0 = Q_2 = (1+|\mathrm{wt}(v_{23})| - h_2 - h_3 - h_4)v_{35} = \lambda_1 v_{35} \tag{5.21}$$

imply $\lambda_1 = 0$. Recall the intertwining operator defined in Lemma 5.2. In these two cases, $T_{5,123}(\xi_{0,1}) = 0$ is equivalent to $T^1|_{V(\omega_1+\omega_2)\otimes V(\mathrm{wt}(\xi_{0,1}))} = 0$. Assume $v^\lambda_{(\omega_1+\omega_2)\otimes(\mathrm{wt}(\xi_{0,1}))}$ is any maximal vector of weight λ appearing in the tensor decomposition $V(\omega_1 + \omega_2) \otimes V(\mathrm{wt}(\xi_{0,1}))$. Then $T^1|_{V(\omega_1+\omega_2)\otimes V(\mathrm{wt}(\xi_{0,1}))} = 0$ iff $T^1(v^\lambda_{(\omega_1+\omega_2)\otimes(\mathrm{wt}(\xi_{0,1}))}) = 0$.

Case 3.3.1 $wt(\xi_{0,1}) = \mathrm{wt}(d_{35}v_\lambda) = (0, \lambda_2 - 1, \lambda_3 + 1, \lambda_4 - 1)$.

Indeed, $T^1(v^\lambda_{(1,1,0,0)\otimes(0,\lambda_2-1,\lambda_3+1,\lambda_4-1)})$ in this case could be written as:

$$\begin{aligned}
& T^1(v^\lambda_{(1,1,0,0)\otimes(0,\lambda_2-1,\lambda_3+1,\lambda_4-1)}) \\
= \;& (T_{4,345}+T_{2,235})\xi_{0,1}+\frac{1}{2}(T_{1,135}-T_{2,235})\xi_{0,1}-\frac{1}{1+\lambda_3}(T_{3,345}-T_{2,245}).(x_4\partial_{x_3})_0.\xi_{0,1} \\
- \;& \frac{1}{2+2\lambda_3}(T_{2,245}-T_{1,145}).(x_4\partial_{x_3})_0.\xi_{0,1}-\frac{3}{1+\lambda_2+\lambda_3}T_{2,345}.x_4\partial_{x_3}.x_3\partial_{x_2}\xi_{0,1} \\
+ \;& \frac{6+3\lambda_3}{(1+\lambda_2+\lambda_3)(1+\lambda_3)}T_{2,345}.(x_3\partial_{x_2})_0.(x_4\partial_{x_3})_0.\xi_{0,1} \\
- \;& \frac{6+3\lambda_3}{(1+\lambda_2+\lambda_3)(1+\lambda_3)}T_{1,345}.(x_2\partial_{x_1})_0.(x_3\partial_{x_2})_0.(x_4\partial_{x_3})_0.\xi_{0,1} \\
+ \;& \frac{3}{1+\lambda_2+\lambda_3}T_{1,345}.(x_4\partial_{x_3})_0.(x_2\partial_{x_1})_0.(x_3\partial_{x_2})_0.\xi_{0,1}=\frac{(2+\lambda_3)(\lambda_2+\lambda_3+7))}{(1+\lambda_3)(\lambda_2+\lambda_3+1))}v_{35}\neq 0. \qquad (5.22)
\end{aligned}$$

Case 3.3.2 $\quad wt(\xi_{0,1})=\mathrm{wt}(d_{45}v_\lambda)=(0,\lambda_2,\lambda_3-1,\lambda_4)$.

Suppose $\lambda_2\neq 0$. Then

$$\begin{aligned}
& T^1(v^\lambda_{(1,1,0,0)\otimes(0,\lambda_2,\lambda_3-1,\lambda_4)})=\frac{-2\lambda_2}{3}(T_{3,345}-T_{2,245})\xi_{0,1}-\frac{\lambda_2}{3}(T_{2,245}-T_{1,145})\xi_{0,1} \\
+ \;& T_{2,345}.(x_3\partial_{x_2})_0.\xi_{0,1}-T_{1,345}.(x_2\partial_{x_1})_0.(x_3\partial_{x_2})_0.\xi_{0,1}=\frac{2\lambda_2(\lambda_2+3)}{3}v_{45}\neq 0 \qquad (5.23)
\end{aligned}$$

induces a contradiction. Assume $\lambda_2=0$. Then it is easily checked that

$$T^1(v^\lambda_{(1,1,0,0)\otimes(0,\lambda_2,\lambda_3-1,\lambda_4)})=[2(T_{3,345}-T_{2,245})+(T_{2,245}-T_{1,145})]\xi_{0,1}=0. \qquad (5.24)$$

Thus $\lambda=(0,0,m,n)$. The proof is complete by Lemma 3.3. $\square$

5.2 Singular vectors of degree two

Theorem 5.4 All the possible degree two singular vectors are listed in the following:

$$\prod_{\mathrm{wt}(d_{12}d_{15})<\sigma\leq\omega_1+\omega_3}\frac{\tilde{c}-\chi_{\sigma+\lambda}(\tilde{c})}{\chi_{\mathrm{wt}(d_{12}d_{15})+\lambda}(\tilde{c})-\chi_{\sigma+\lambda}(\tilde{c})}.d_{12}d_{15}v_\lambda,\ \text{where}\ \ \lambda=(m,0,0,1),\ m\in\mathbb{N}.$$

Proof The leading term of any singular vector of degree two could be written as:

$$\xi_{0,2}=\sum_{j\in\overline{1,35},k\in 1,\mathrm{mult}(\overrightarrow{w}_j^{\omega_1+\omega_3})}v_{j,k}^{\omega_1+\omega_3}v^\lambda_{j,k},\ v^\lambda_{j,k}\in V(\lambda), \qquad (5.27)$$

which should satisfy $T_{5,123}.\xi_{0,2}=0$. Assume

$$T_{5,123}.\xi_{0,2}=\sum_{1\leq i<j\leq 5}d_{ij}t_{ij},t_{ij}\in V(\lambda). \qquad (5.28)$$

Then we could derive the following equations:

$$t_{12}=E_{52}v^\lambda_{1,1}-E_{51}v^\lambda_{2,1}=0,t_{15}=E_{53}v^\lambda_{5,1}-E_{52}v^\lambda_{10,1}+E_{51}(v^\lambda_{11,2}+v^\lambda_{11,3})+v^\lambda_{1,1}=0,$$

$$t_{13}=E_{53}v^\lambda_{1,1}-E_{51}v^\lambda_{6,1}=0,t_{25}=E_{53}v^\lambda_{9,1}+E_{52}(v^\lambda_{11,1}+v^\lambda_{11,3})+E_{51}v^\lambda_{19,1}+v^\lambda_{2,1}=0,$$

$$t_{23}=E_{53}v^\lambda_{2,1}-E_{52}v^\lambda_{6,1}=0,t_{35}=E_{53}(v^\lambda_{11,1}+v^\lambda_{11,2})-E_{52}v^\lambda_{17,1}+E_{51}v^\lambda_{22,1}+v^\lambda_{6,1}=0. \qquad (5.29)$$

It follows from $v_{1,1}^{\lambda} \neq 0$ that one of $v_{5,1}^{\lambda},\ v_{10,1}^{\lambda},\ v_{11,2}^{\lambda} + v_{11,3}^{\lambda}$ should be nonzero. Hence, the information of the weights in Table 5 implies that $\mathrm{wt}(\xi_{0,2})$ should be restricted to the following cases:

$$\begin{aligned}&\mathrm{wt}(\xi_{0,2}) \in \{\lambda + \overrightarrow{w}_5^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_9^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{10}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{11}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{15}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{16}^{\omega_1+\omega_3},\\ &\lambda + \overrightarrow{w}_{17}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{19}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{21}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{22}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_i^{\omega_1+\omega_3} (i \in \overline{24,35})\}\end{aligned} \tag{5.30}$$

Case 1 $\quad \mathrm{wt}(\xi_{0,2}) \in \{\lambda + \overrightarrow{w}_5^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{10}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{15}^{\omega_1+\omega_3}\}$

In these three cases, $v_{2,1}^{\lambda} = v_{6,1}^{\lambda} = 0$.

Case 1.1 $\quad wt(\xi_{0,2}) = \lambda + \overrightarrow{w}_5^{\omega_1+\omega_3}$

We have $wt(v_{1,1}^{\lambda}) = \lambda - \alpha_3 - \alpha_4$ and $\lambda_4 > 0$. So $0 = E_{25}.t_{12} = E_{25}.E_{52}v_{1,1}^{\lambda} = (h_2 + h_3 + h_4)v_{1,1}^{\lambda} = (\lambda_2 + \lambda_3 + \lambda_4 - 1)v_{1,1}^{\lambda}$ yields $(\lambda_2, \lambda_3, \lambda_4) = (0, 0, 1)$. Then $\chi_{\overrightarrow{w}_1^{\omega_1+\omega_3}}(\tilde{c}) = \frac{3\lambda_1+2}{25}$ by Lemma 3.5. And

$$T_{5,123}.\xi_{0,2} = T_{5,123}.(\tilde{c} - \chi_{\overrightarrow{w}_1^{\omega_1+\omega_3}}(\tilde{c})).d_{12}d_{15}.v_\lambda = T_{5,123}.\tilde{c}.d_{12}d_{15}.v_\lambda - \chi_{\overrightarrow{w}_1^{\omega_1+\omega_3}}(\tilde{c})d_{15}E_{53}.v_\lambda = 0, \tag{5.31}$$

since

$$T_{5,123}.\tilde{c}.d_{12}d_{15}.v_\lambda \overset{\text{by (5.13)}}{=} Q_{12}^1 d_{15}v_\lambda - Q_{15}^1 d_{12}v_\lambda;$$

where

$$\begin{aligned} Q_{12}^1 &\overset{\text{by(5.3)}}{=} Q_{12}^0(\tilde{c} + \frac{h_2^*}{10}) + \frac{1}{10}Q_{13}^0 E_{23} - \frac{1}{10}Q_{23}^0 E_{13},\ Q_{15}^1 \overset{\text{by(5.3)}}{=} \frac{1}{10}(Q_{12}^0 E_{52} + Q_{13}^0 E_{53}),\\ Q_{12}^0 &= E_{53} + \frac{(x_5\partial_{x_3})_0'}{2},\ Q_{13}^0 = -E_{52} - \frac{(x_5\partial_{x_2})_0'}{2}, Q_{23}^0 = E_{51} + \frac{(x_5\partial_{x_1})_0'}{2},\\ \tilde{c}.d_{15}v_\lambda &\overset{\text{by(5.1)}}{=} \frac{1}{10}[d_{15}(h_1^* - h_4^*).v_\lambda + \sum_{i=2}^{4} d_{1i}E_{5i}v_\lambda] = \frac{1}{10}[\frac{3\lambda_1 - 3}{5}d_{15}v_\lambda + \sum_{i=2}^{4} d_{1i}E_{5i}v_\lambda]. \end{aligned} \tag{5.32}$$

That is, $\lambda = (m, 0, 0, 1)$.

Case 1.2 $\quad wt(\xi_{0,2}) = \lambda + \overrightarrow{w}_{10}^{\omega_1+\omega_3}$

We have $\mathrm{wt}(v_{1,1}^{\lambda}) = \lambda - \alpha_2 - \alpha_3 - \alpha_4$ and $\lambda_2 > 0, \lambda_4 > 0$, $E_{25}.v_{1,1}^{\lambda} = -v_{10,1}^{\lambda}$. So

$$0 = E_{25}t_{12} = (h_2 + h_3 + h_4)v_{1,1}^{\lambda} + E_{52}.E_{25}.v_{1,1}^{\lambda} = (\lambda_2 + \lambda_3 + \lambda_4 - 2)v_{1,1}^{\lambda} - E_{52}.v_{10,1}^{\lambda}, \tag{5.33}$$

$$0 = E_{25}^2 t_{12} = E_{25}.[(\lambda_2 + \lambda_3 + \lambda_4 - 2)v_{1,1}^{\lambda} - E_{52}.v_{10,1}^{\lambda}] = -2(\lambda_2 + \lambda_3 + \lambda_4 - 1)v_{10,1}^{\lambda} \tag{5.34}$$

yields a contradiction.

Case 1.3 $\quad wt(\xi_{0,2}) = \lambda + \overrightarrow{w}_{15}^{\omega_1+\omega_3}$

We have $\mathrm{wt}(v_{10,1}^{\lambda}) = \lambda - \alpha_3$ and $\lambda_3 > 0$, $E_{23}.v_{1,5}^{\lambda} = -v_{10,1}^{\lambda}$. Then

$$E_{53}v_{5,1}^{\lambda} - E_{52}v_{10,1}^{\lambda} + v_{1,1}^{\lambda} = 0, \tag{5.35}$$

$$0 = E_{25}(E_{53}v_{5,1}^{\lambda} - E_{52}v_{10,1}^{\lambda} + v_{1,1}^{\lambda}) = E_{23}v_{5,1}^{\lambda} - (h_2 + h_3 + h_4)v_{10,1}^{\lambda} + E_{25}v_{1,1}^{\lambda} \tag{5.36}$$

imply $(\lambda_2, \lambda_3, \lambda_4) = (0, 0, 0)$. A contradiction arises.

Case 2 $\quad \mathrm{wt}(\xi_{0,2}) \in \{\lambda + \overrightarrow{w}_9^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{11}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{16}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{17}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{19}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{21}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{22}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_i^{\omega_1+\omega_3} (i \in \overline{24,35})\}$

Case 2.1 $\quad wt(\xi_{0,2}) = \lambda + \overrightarrow{w}_9^{\omega_1+\omega_3}$

We have $v_{2,1}^\lambda \neq 0$, since $E_{35}v_{2,1}^\lambda = -v_{9,1}^\lambda$. Note that $v_{6,1}^\lambda = 0$, $E_{15}v_{1,1}^\lambda = E_{15}v_{2,1}^\lambda = 0$, $E_{12}v_{1,1}^\lambda = -v_{2,1}^\lambda$, $\mathrm{wt}(v_{2,1}^\lambda) = \lambda - \alpha_3 - \alpha_4$ and $\lambda_1 > 0, \lambda_4 > 0$. Then

$$0 = E_{15}.t_{12} = E_{15}.(E_{52}v_{1,1}^\lambda - E_{51}v_{2,1}^\lambda) = (E_{12} + E_{52}E_{15})v_{1,1}^\lambda - (\sum_{i=1}^{4} h_i + E_{51}E_{15})v_{2,1}^\lambda \tag{5.37}$$

implies $|\lambda| = 0$. A contradiction arises.

Case 2.2 $\quad \mathrm{wt}(\xi_{0,2}) \in \{\lambda + \overrightarrow{w}_{11}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{16}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{17}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{19}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{21}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{22}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_i^{\omega_1+\omega_3} (i \in \overline{24, 35})\}$

Case 2.2.1 $\quad \mathrm{wt}(\xi_{0,2}) \in \{\lambda + \overrightarrow{w}_{11}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{16}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{17}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{19}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{21}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{22}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{24}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{26}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{27}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{29}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{30}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{33}^{\omega_1+\omega_3}.\}$

In these cases, $v_{25,1}^\lambda = v_{28,1}^\lambda = 0$. Assume $v_{11,2}^\lambda + v_{11,3}^\lambda \neq 0$. Then

$$0 = E_{15}t_{15} \overset{\text{by}(5.29)}{=} (2 + \sum_{i=1}^{4} h_i)(v_{11,2}^\lambda + v_{11,3}^\lambda) = 0 \tag{5.38}$$

yields a contradiction. Hence, $v_{11,2}^\lambda + v_{11,3}^\lambda = 0$. Furthermore, either the assertion $v_{10,1}^\lambda \neq 0,\ (h_2 + h_3 + h_4)v_{10,1}^\lambda = 0$ or the assertion $v_{10,1}^\lambda = 0,\ v_{5,1}^\lambda \neq 0,\ (h_3 + h_4 - 1)v_{5,1}^\lambda = 0$ holds. By detailed check case by case, only the cases $\mathrm{wt}(\xi_{0,2}) \in \{\lambda + \overrightarrow{w}_{16}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{24}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{26}^{\omega_1+\omega_3}\}$ satisfy this assertion.

For the case $\mathrm{wt}(\xi_{0,2}) = \lambda + \overrightarrow{w}_{16}^{\omega_1+\omega_3}$, we get $v_{10,1}^\lambda = 0,\ v_{5,1}^\lambda \neq 0,\ (\lambda_3, \lambda_4) = (1, 0)$. And we could write $\xi_{0,2} = y_{12}\xi'_{0,2}$, where $\xi'_{0,2} = \sum\limits_{1 \le i < j \le 5} d_{ij}v_{ij}$. Hence, $T_{5,123}.\xi_{0,2} = T_{5,123}.y_{12}\xi'_{0,2} = ([T_{5,123}, y_{12}] + y_{12}T_{5,123}).\xi'_{0,2} = (x_5\partial_{x_3} + \frac{1}{2}(-1)^{|34,45|}y_{45}\partial_{34}).\xi'_{0,2} = \frac{1}{2}(-1)^{|34,45|}y_{45}\partial_{34}.\xi'_{0,2} \neq 0$.

For the case $\mathrm{wt}(\xi_{0,2}) = \lambda + \overrightarrow{w}_{24}^{\omega_1+\omega_3}$, we get $v_{19,1}^\lambda \neq 0$. The equation $E_{15}t_{25} = 0$ implies that

$$0 = E_{15}v_{2,1}^\lambda + (E_{13} + E_{53}E_{15})v_{9,1}^\lambda + (E_{12} + E_{52}E_{15})(v_{11,1}^\lambda + v_{11,3}^\lambda) + \sum_{i=1}^{4} h_i v_{19,1}^\lambda = (3 + |\lambda|)v_{19,1}^\lambda. \tag{5.39}$$

For the case $\mathrm{wt}(\xi_{0,2}) = \lambda + \overrightarrow{w}_{26}^{\omega_1+\omega_3}$, one of $v_{19,1}^\lambda$ and $v_{22,1}^\lambda$ should be nonzero, otherwise $v_{26,i}^\lambda = 0$. Then, the equation $E_{15}t_{35} = 0$ implies that

$$0 = E_{15}v_{6,1}^\lambda + (E_{13} + E_{53}E_{15})(v_{11,1}^\lambda + v_{11,2}^\lambda) - (E_{12} + E_{52}E_{15})v_{17,1}^\lambda) + \sum_{i=1}^{4} h_i.v_{22,1}^\lambda = (1 + |\lambda|)v_{22,1}^\lambda. \tag{5.40}$$

Case 2.2.2 $\quad \mathrm{wt}(\xi_{0,2}) = \lambda + \overrightarrow{w}_{25}^{\omega_1+\omega_3}$

In this case, $v_{25,1}^\lambda \neq 0$, $v_{28,1}^\lambda = 0$ and $wt(\xi_{0,2}) = \lambda + \mathrm{wt}(d_{15}d_{25})$. Then

$$0 = E_{15}t_{15} = (2 + \sum_{i=1}^{4} h_i)(v_{11,2}^\lambda - v_{11,3}^\lambda) - E_{53}v_{25,1}^\lambda,$$

$$E_{35}.E_{15}t_{15} = (2 + |\mathrm{wt}(\overrightarrow{w}_{25}^{\omega_1+\omega_3})| - h_3 - h_4)v_{25,1}^\lambda = 0. \tag{5.41}$$

So $\lambda_1 + \lambda_2 + 1 = 0$. A contradiction arises.

Case 2.2.3 $\quad \mathrm{wt}(\xi_{0,2}) \in \{\lambda + \overrightarrow{w}_{28}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{31}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{32}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{34}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{35}^{\omega_1+\omega_3}, \}$

In these cases, $v_{25,1}^\lambda \neq 0$ and $v_{28,1}^\lambda \neq 0$.

Case 2.2.3.1 $\quad \mathrm{wt}(\xi_{0,2}) \in \{\lambda + \overrightarrow{w}_{28}^{\omega_1+\omega_3}, \lambda + \overrightarrow{w}_{31}^{\omega_1+\omega_3}\}$

The equations

$$0=E_{15}t_{12}=(E_{12}+E_{52}E_{15})v^{\lambda}_{1,1}-(\sum_{i=1}^{4}h_i+E_{51}E_{15})v^{\lambda}_{2,1}=-(1+|\mathrm{wt}(\overrightarrow{w}^{\omega_1+\omega_3}_{2})|)v^{\lambda}_{1,1}+E_{52}(v^{\lambda}_{11,2}+v^{\lambda}_{11,3}), \tag{5.42}$$

$$E^2_{25}E_{15}t_{12}=2(\lambda_2+\lambda_3+\lambda_4-1)v^{\lambda}_{28,1}=0 \tag{5.43}$$

induce a contradiction.

Case 2.2.3.2 $\quad \mathrm{wt}(\xi_{0,2})\in\{\lambda+\overrightarrow{w}^{\omega_1+\omega_3}_{32},\lambda+\overrightarrow{w}^{\omega_1+\omega_3}_{34},\lambda+\overrightarrow{w}^{\omega_1+\omega_3}_{35}\}$

First, we have $v^{\lambda}_{32,1}\neq 0$ in these cases, since $E_{34}v^{\lambda}_{32,1}=-v^{\lambda}_{34,1},\ E_{24}v^{\lambda}_{32,1}=v^{\lambda}_{35,1}$. Then

$$\begin{aligned}&0=E_{15}t_{35}=E_{15}v^{\lambda}_{6,1}+(E_{13}+E_{53}E_{15})(v^{\lambda}_{11,1}+v^{\lambda}_{11,2})-(E_{12}+E_{52}E_{15})v^{\lambda}_{17,1}\\&+(\sum_{i=1}^{4}h_i)v^{\lambda}_{22,1}=(1+|\mathrm{wt}(\overrightarrow{w}^{\omega_1+\omega_3}_{22})|)v^{\lambda}_{22,1}+E_{53}v^{\lambda}_{32,1},\end{aligned}$$

$$E_{35}E_{15}t_{35}=(-1-|\mathrm{wt}(\overrightarrow{w}^{\omega_1+\omega_3}_{22})|+h_3+h_4)v^{\lambda}_{32,1}=0. \tag{5.44}$$

If $\mathrm{wt}(\xi_{0,2})=\lambda+\overrightarrow{w}^{\omega_1+\omega_3}_{32}$, then $\mathrm{wt}(\overrightarrow{w}^{\omega_1+\omega_3}_{32})=\lambda$ and $\mathrm{wt}(\overrightarrow{w}^{\omega_1+\omega_3}_{22})=\lambda-\alpha_3-\alpha_4$. If $\mathrm{wt}(\xi_{0,2})=\lambda+\overrightarrow{w}^{\omega_1+\omega_3}_{34}$, then $\mathrm{wt}(\overrightarrow{w}^{\omega_1+\omega_3}_{32})=\lambda-\alpha_3$ and $\mathrm{wt}(\overrightarrow{w}^{\omega_1+\omega_3}_{22})=\lambda-2\alpha_3-\alpha_4$. If $\mathrm{wt}(\xi_{0,2})=\lambda+\overrightarrow{w}^{\omega_1+\omega_3}_{35}$, then $\mathrm{wt}(\overrightarrow{w}^{\omega_1+\omega_3}_{32})=\lambda-\alpha_2-\alpha_3$ and $\mathrm{wt}(\overrightarrow{w}^{\omega_1+\omega_3}_{22})=\lambda-\alpha_2-2\alpha_3-\alpha_4$. Thus, (5.44) yields $\lambda_1+\lambda_2=0$ or $\lambda_1+\lambda_2+1=0$. A contradiction arises. $\square$

5.3 Singular vectors of degree three

Theorem 5.5 All the possible degree three singular vectors are listed in the following:

$$e^{-\frac{P}{2}}.\prod_{\mathrm{wt}(d_{15}d_{25}d_{45})<\sigma\leq 2\omega_1+\omega_4}\frac{\tilde{c}-\chi_{\sigma+\lambda}(\tilde{c})}{\chi_{\mathrm{wt}(d_{15}d_{25}d_{45})+\lambda}(\tilde{c})-\chi_{\sigma+\lambda}(\tilde{c})}.d_{15}d_{25}d_{45}v_{\lambda},$$

$$\text{where}\ \ \lambda=(0,0,m,n),\ 1\leq m\in\mathbb{N},\ 2\leq n\in\mathbb{N}.$$

Proof The leading term of any singular vector of degree three could be written as:

$$\xi_{0,3}=\sum_{j\in\overline{1,55},k\in\overline{1,\mathrm{mult}(\overrightarrow{w}^{2\omega_1+\omega_4}_{j})}}v^{2\omega_1+\omega_4}_{j,k}v^{\lambda}_{j,k},\ \ v^{\lambda}_{j,k}\in V(\lambda), \tag{5.45}$$

which should satisfy $T_{5,123}.\xi_{0,3}=0$ and $(x_i\partial_{x_j})_0.\xi_{0,3}=0$ ($1\leq i<j\leq 5$). Assume

$$T_{5,123}.\xi_{0,3}=\sum_{i_1<j_1,i_2<j_2,(i_1,j_1)\neq(i_2,j_2)}d_{i_1j_1}d_{i_2j_2}t_{i_1j_1,i_2j_2},\ \ t_{i_1j_1,i_2j_2}\in V(\lambda). \tag{5.46}$$

Since $v^{\lambda}_{1,1}\neq 0$, the equation

$$0=t_{14,15}=-v^{\lambda}_{1,1}+E_{53}v^{\lambda}_{7,1}-E_{52}v^{\lambda}_{13,1}+E_{51}(v^{\lambda}_{18,1}-v^{\lambda}_{18,4}) \tag{5.47}$$

implies that one of the terms $v^{\lambda}_{7,1}$, $v^{\lambda}_{13,1}$ and $v^{\lambda}_{18,1}-v^{\lambda}_{18,4}$ should be nonzero. Hence, the information of the weights of Table 6-7 induces that $wt(\xi_{0,3})$ could be restricted to the following cases:

$$\begin{aligned}&wt(\xi_{0,3})\in\{\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{7},\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{12},\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{13},\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{17},\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{18},\\&\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{21},\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{22},\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{23},\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{24},\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{i}(i\in\overline{26,55})\}.\end{aligned} \tag{5.48}$$

Case 1 $\quad wt(\xi_{0,3}) = \lambda + \overrightarrow{w}_7^{2\omega_1+\omega_4}$

In this case, $v_{6,1}^\lambda = 0$, $v_{3,1}^\lambda \neq 0$, $wt(v_{3,1}^\lambda) = \lambda - \alpha_3$, $E_{34}v_{3,1}^\lambda = -v_{7,1}^\lambda$. Then

$$E_{25}t_{12,15} = E_{25}(E_{52}v_{3,1}^\lambda - E_{51}v_{6,1}^\lambda) = (h_2 + h_3 + h_4)v_{3,1}^\lambda = (\lambda_2 + \lambda_3 + \lambda_4)v_{3,1}^\lambda = 0 \tag{5.49}$$

contradicts $\lambda_3 > 0$.

Case 2 $\quad wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}_{12}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{17}^{2\omega_1+\omega_4}\}$

Note that $E_{34}v_{6,1}^\lambda = -v_{12,1}^\lambda, E_{12}v_{12,1}^\lambda = -v_{17,1}^\lambda, E_{12}v_{3,1}^\lambda = -2v_{6,1}^\lambda$. Hence, $v_{3,1}^\lambda \neq 0$ and $v_{6,1}^\lambda, \neq 0$. Since $wt(v_{6,1}^\lambda) = \lambda - \alpha_3$ for $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}_{12}^{2\omega_1+\omega_4}$; $wt(v_{6,1}^\lambda) = \lambda - \alpha_1 - \alpha_3$ for $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}_{17}^{2\omega_1+\omega_4}$. The equation $E_{15}t_{12,15} = -(2 + |\mathrm{wt}v_{6,1}^\lambda|)v_{6,1}^\lambda = 0$ induces that $|\lambda| < 0$ in both cases.

Case 3 $\quad wt(\xi_{0,3}) = \lambda + \overrightarrow{w}_{13}^{2\omega_1+\omega_4}$

We have $E_{25}v_{1,1}^\lambda = -v_{13,1}^\lambda$, $E_{23}v_{7,1}^\lambda = -v_{13,1}^\lambda$. Then

$$0 = E_{25}t_{14,15} \overset{\text{by}(5.47)}{=} -E_{25}v_{1,1}^\lambda + E_{23}v_{7,1}^\lambda - \sum_{i=2}^{4} h_i v_{13,1}^\lambda = -(\lambda_2 + \lambda_3 + \lambda_4)v_{13,1}^\lambda = 0, \tag{5.50}$$

contradicts $\lambda_2 > 0$.

Case 4 $\quad wt(\xi_{0,3}) = \lambda + \overrightarrow{w}_{22}^{2\omega_1+\omega_4}$

We have $v_{11,1}^\lambda \neq 0$, since $E_{24}v_{11,1}^\lambda = v_{22,1}^\lambda$. Then

$$0 = E_{15}t_{13,15} = E_{15}(E_{53}v_{3,1}^\lambda - E_{51}v_{11,1}^\lambda) = -(2 + |\lambda|)v_{11,1}^\lambda \tag{5.51}$$

yields a contradiction.

Case 5 $\quad wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}_{24}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{28}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{30}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{35}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{36}^{2\omega_1+\omega_4}\}$

For $wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}_{24}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{30}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{36}^{2\omega_1+\omega_4}\}$, we have $v_{28,1}^\lambda = 0$. Then

$$0 = E_{25}t_{15,25} = E_{25}(E_{52}v_{24,1}^\lambda - E_{51}v_{28,1}^\lambda) = (h_2 + h_3 + h_4)v_{24,1}^\lambda \tag{5.52}$$

yields $\lambda_2 + \lambda_3 + \lambda_4 = 0$ or 1. For $wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}_{28}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{35}^{2\omega_1+\omega_4}\}$, the equation

$$0 = E_{15}t_{15,25} = -(h_1 + h_2 + h_3 + h_4)v_{28,1}^\lambda + (E_{12} + E_{52}E_{15})v_{24,1}^\lambda \tag{5.53}$$

yields $1 + |\lambda| = 0$.

Case 6 $\quad wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}_{23}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{29}^{2\omega_1+\omega_4}\}$

We have $E_{35}v_{10,1}^\lambda = v_{23,1}^\lambda, E_{25}v_{10,1}^\lambda = -v_{29,1}^\lambda, E_{23}v_{23,1}^\lambda = -v_{29,1}^\lambda$. Consider the equation

$$t_{14,45} = -v_{10,1}^\lambda + E_{53}v_{23,1}^\lambda - E_{52}v_{29,1}^\lambda - E_{51}(v_{32,1}^\lambda - v_{32,4}^\lambda). \tag{5.54}$$

For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}_{23}^{2\omega_1+\omega_4}$, $E_{35}t_{14,45} = (\lambda_3 + \lambda_4 - 1)v_{23,1}^\lambda = 0$ induces $\lambda_3 + \lambda_4 = 1$, which contradicts $\lambda_3 > 1$. For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}_{29}^{2\omega_1+\omega_4}$, we have $E_{25}t_{14,45} = -(\lambda_2 + \lambda_3 + \lambda_4)v_{29,1}^\lambda = 0$ induces $\lambda_2 + \lambda_3 + \lambda_4 = 0$, which contradicts $\lambda_2 > 0, \lambda_3 > 0$.

Case 7 $\quad wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}_{27}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{39}^{2\omega_1+\omega_4}, \lambda + \overrightarrow{w}_{42}^{2\omega_1+\omega_4}\}$

Note that in these cases, the equations are derived: $E_{35}v^{\lambda}_{15,1} = v^{\lambda}_{27,1}, E_{15}v^{\lambda}_{15,1} = v^{\lambda}_{39,1}, E_{13}v^{\lambda}_{27,1} = v^{\lambda}_{39,1}, E_{12}v^{\lambda}_{32,4} = -v^{\lambda}_{39,1}, E_{12}v^{\lambda}_{32,1} = 0 = E_{12}v^{\lambda}_{32,3}, E_{25}v^{\lambda}_{15,1} = -\sum_{i=1}^{4} v^{\lambda}_{32,i}, E_{23}v^{\lambda}_{27,1} = -v^{\lambda}_{32,1} - 2v^{\lambda}_{32,3} - v^{\lambda}_{32,4}, E_{34}v^{\lambda}_{32,1} = -v^{\lambda}_{42,1}, E_{34}v^{\lambda}_{32,2} = v^{\lambda}_{42,1}, E_{34}v^{\lambda}_{32,3} = 0 = E_{34}v^{\lambda}_{32,4}$. Consider the equation

$$t_{24,45} = -v^{\lambda}_{15,1} + E_{53}v^{\lambda}_{27,1} - E_{52}(v^{\lambda}_{32,1} + v^{\lambda}_{32,3} + v^{\lambda}_{32,4}) + E_{51}v^{\lambda}_{39,1}. \tag{5.55}$$

For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{27}$, $E_{35}t_{24,45} = (\lambda_3 + \lambda_4 - 1)v^{\lambda}_{27,1} = 0$ induces $\lambda_3 + \lambda_4 = 1$, which contradicts $\lambda_3 > 1$. For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{39}$, $E_{15}t_{24,45} = (1 + |\lambda|)v^{\lambda}_{39,1} = 0$ induces $1 + |\lambda| = 0$. For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{42}$, Then $E_{15}t_{24,45} = v^{\lambda}_{32,2} - v^{\lambda}_{32,3} - (\lambda_2 + \lambda_3 + \lambda_4)(v^{\lambda}_{32,1} + v^{\lambda}_{32,3} + v^{\lambda}_{32,4}) = 0$, and $E_{34}E_{15}t_{24,45} = (1 + \lambda_2 + \lambda_3 + \lambda_4)v^{\lambda}_{42,1} = 0$ induces $1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$.

Case 8 $\quad wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{31}, \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{33}\}$

Observe these equations are derived: $E_{15}v^{\lambda}_{4,1} = v^{\lambda}_{31,1}, E_{13}v^{\lambda}_{17,1} = v^{\lambda}_{31,1}, E_{12}v^{\lambda}_{21,1} = -v^{\lambda}_{31,1}, E_{12}v^{\lambda}_{21,2} = v^{\lambda}_{31,1}, E_{12}v^{\lambda}_{21,3} = 0, E_{12}v^{\lambda}_{21,4} = 0$. Consider the equation

$$t_{24,25} = -v^{\lambda}_{4,1} + E_{53}v^{\lambda}_{17,1} - E_{52}(2v^{\lambda}_{21,1} + v^{\lambda}_{21,2} + v^{\lambda}_{21,3}) + E_{51}v^{\lambda}_{31,1}. \tag{5.56}$$

It follows from $E_{15}t_{24,25} = 0$ that $(1 + |\lambda|)v^{\lambda}_{31,1} = 0$.

Case 9 $\quad wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{37}, \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{38}\}$

Note $E_{15}v^{\lambda}_{14,1} = v^{\lambda}_{38,1}, E_{13}v^{\lambda}_{26,2} = v^{\lambda}_{38,1}, E_{12}v^{\lambda}_{37,1} = -v^{\lambda}_{38,1}, E_{25}v^{\lambda}_{14,1} = -v^{\lambda}_{37,1}, E_{23}v^{\lambda}_{26,2} = -v^{\lambda}_{37,1}$. Consider the equation

$$t_{34,35} = -v^{\lambda}_{14,1} + E_{53}v^{\lambda}_{26,2} - E_{52}v^{\lambda}_{37,1} + E_{51}v^{\lambda}_{38,1}. \tag{5.57}$$

For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{37}$, $E_{25}t_{34,35} = -(\lambda_2 + \lambda_3 + \lambda_4)v^{\lambda}_{37,1} = 0$ implies $\lambda_2 + \lambda_3 + \lambda_4 = 0$, which which contradicts $\lambda_2 > 2$. For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{38}$, $E_{15}t_{34,35} = (1 + |\lambda|)v^{\lambda}_{38,1} = 0$.

Case 10 $\quad wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{40}, \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{44}, \overrightarrow{w}^{2\omega_1+\omega_4}_{48}, \overrightarrow{w}^{2\omega_1+\omega_4}_{51}\}$

Note $E_{25}v^{\lambda}_{19,1} = -v^{\lambda}_{40,1}, E_{23}v^{\lambda}_{32,3} = -v^{\lambda}_{40,1}, E_{15}v^{\lambda}_{19,1} = v^{\lambda}_{44,1}, E_{13}v^{\lambda}_{32,3} = v^{\lambda}_{44,1}, E_{12}v^{\lambda}_{40,1} = -v^{\lambda}_{44,1}$. Consider the equation

$$t_{34,45} = -v^{\lambda}_{19,1} + E_{53}v^{\lambda}_{32,3} - E_{52}v^{\lambda}_{40,1} + E_{51}v^{\lambda}_{44,1}, \tag{5.58}$$

For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{40}, \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{48}$, the equation $E_{25}t_{34,45} = 0$ implies $-(\lambda_2 + \lambda_3 + \lambda_4)v^{\lambda}_{40,1} = 0$. For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{44}, \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{51}$, the equation $E_{15}t_{34,45} = 0$ implies $(1 + |\lambda|)v^{\lambda}_{44,1} = 0$.

Case 11 $\quad wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{45}, \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{52}\}$

Note $E_{15}v^{\lambda}_{21,3} = -v^{\lambda}_{45,1}, E_{15}v^{\lambda}_{21,4} = v^{\lambda}_{45,1}, E_{13}v^{\lambda}_{35,1} = v^{\lambda}_{45,1}, E_{12}v^{\lambda}_{41,1} = -v^{\lambda}_{45,1}, E_{12}v^{\lambda}_{41,i} = 0 \ (i \in \overline{2,4}), E_{15}v^{\lambda}_{18,3} = v^{\lambda}_{41,1} - v^{\lambda}_{41,4}, E_{15}v^{\lambda}_{18,4} = -v^{\lambda}_{41,1} - v^{\lambda}_{41,3}, E_{13}v^{\lambda}_{30,1} = v^{\lambda}_{41,1} - v^{\lambda}_{41,2} - v^{\lambda}_{41,3}, E_{12}v^{\lambda}_{36,1} = -2v^{\lambda}_{41,1} - v^{\lambda}_{41,2}, E_{15}v^{\lambda}_{30,1} = E_{35}v^{\lambda}_{41,3} = -E_{35}v^{\lambda}_{41,4} = -v^{\lambda}_{52,1}$. Consider the equation

$$t_{25,45} = -v^{\lambda}_{21,3} - v^{\lambda}_{21,4} + E_{53}v^{\lambda}_{35,1} - E_{52}(v^{\lambda}_{41,1} + v^{\lambda}_{41,2} + v^{\lambda}_{41,3}) + E_{51}v^{\lambda}_{45,1} = 0. \tag{5.59}$$

For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{45}$, the equation $E_{15}t_{25,45} = 0$ implies $(2 + |\lambda|)v^{\lambda}_{45,1} = 0$. Now suppose $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{52}$. The equation $t_{15,45} = 0$ implies that

$$E_{15}t_{15,45} = 0 = (|\lambda| + 2)v^{\lambda}_{41,1} + (1 - |\lambda|)v^{\lambda}_{41,3} + v^{\lambda}_{41,4} - E_{53}v^{\lambda}_{52,1},$$

$$0 = E_{35}E_{15}t_{15,45} = (\lambda_1 + \lambda_2)v^{\lambda}_{52,1}, \tag{5.60}$$

i.e. $\lambda_1 = \lambda_2 = 0$. Then $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{52} = (0, 1, \lambda_3 - 1, \lambda_4 - 2)$. Observe that

$$\Pi((\omega_1 + \omega_2) \otimes \mathrm{wt}(\xi_{0,3})) \bigcap \Pi((\omega_1 + \omega_3) \otimes \lambda) = \{\mathrm{wt}(\xi_{0,3}) + (0, 0, 1, 0), \mathrm{wt}(\xi_{0,3}) + (1, -1, 0, 1)\}. \tag{5.61}$$

$$\Pi((\omega_1+\omega_2)\otimes(\mathrm{wt}(\xi_{0,3})+(0,0,1,0)))\bigcap\{\lambda\} = \emptyset,\ \Pi((\omega_1+\omega_2)\otimes(\mathrm{wt}(\xi_{0,3})+(1,-1,0,1)))\bigcap\{\lambda\} = \emptyset. \tag{5.62}$$

Then (5.62) implies that

$$T^2|_{V(\omega_1+\omega_2)\otimes V(\mathrm{wt}(\xi_{0,3})+(0,0,1,0))} = 0, T^2|_{V(\omega_1+\omega_2)\otimes V(\mathrm{wt}(\xi_{0,3})+(1,-1,0,1))} = 0. \tag{5.63}$$

Note that $T_{5,123}\xi_{0,3} = 0$ iff $T^3|_{V(\omega_1+\omega_2)\otimes V(\mathrm{wt}(\xi_{0,3}))} = 0$ by Lemma 5.2. Assume $v^{(0,1,\lambda_3,\lambda_4-2)}_{(\omega_1+\omega_2)\otimes\mathrm{wt}(\xi_{0,3})}$ (resp. $v^{(1,0,\lambda_3-1,\lambda_4-1)}_{(\omega_1+\omega_2)\otimes\mathrm{wt}(\xi_{0,3})}$) is any maximal vector of weight $(0, 1, \lambda_3, \lambda_4 - 2)$ (resp. $(1, 0, \lambda_3 - 1, \lambda_4 - 1)$) appearing in the tensor decomposition $V(\omega_1 + \omega_2) \otimes V(\mathrm{wt}(\xi_{0,3}))$. Since $T^3 V(\omega_1 + \omega_2) \otimes V(\mathrm{wt}(\xi_{0,3})) \subseteq V(\omega_1 + \omega_3) \otimes V(\lambda)$, $T^3|_{V(\omega_1+\omega_2)\otimes V(\mathrm{wt}(\xi_{0,3}))} = 0$ iff $T^3.(v^{(0,1,\lambda_3,\lambda_4-2)}_{(\omega_1+\omega_2)\otimes\mathrm{wt}(\xi_{0,3})}) = 0, T^3(v^{(1,0,\lambda_3-1,\lambda_4-1)}_{(\omega_1+\omega_2)\otimes\mathrm{wt}(\xi_{0,3})}) = 0$. Otherwise, we could get a singular vector of degree two with weights $(0, 1, \lambda_3, \lambda_4 - 2)$ and $(1, 0, \lambda_3 - 1, \lambda_4 - 1)$ respectively by (5.63), which contradicts proposition 5.3. Hence, we get $\lambda = (0, 0, m, n)$.

Case 12 $\quad wt(\xi_{0,3}) \in \{\lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{43}, \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{47}\}$

Consider the equation

$$t_{15,45} = -v^{\lambda}_{18,3} - v^{\lambda}_{18,4} + E_{53}v^{\lambda}_{30,1} - E_{52}v^{\lambda}_{36,1} + E_{51}v^{\lambda}_{41,1} - E_{51}v^{\lambda}_{41,3}, \tag{5.64}$$

Note $E_{15}v^{\lambda}_{18,3} = v^{\lambda}_{41,1} - v^{\lambda}_{41,4}, E_{15}v^{\lambda}_{18,4} = -v^{\lambda}_{41,1} - v^{\lambda}_{41,3}, E_{13}v^{\lambda}_{30,1} = v^{\lambda}_{41,1} - v^{\lambda}_{41,2} - v^{\lambda}_{41,3}, E_{12}v^{\lambda}_{36,1} = -2v^{\lambda}_{41,1} - v^{\lambda}_{41,2}$. Then $E_{15}t_{15,45} = 0$ induces that $3v^{\lambda}_{41,1} + v^{\lambda}_{41,4} + \sum_{i=1}^{4} h_i(v^{\lambda}_{41,1} - v^{\lambda}_{41,3}) = 0$. Since $E_{45}v^{\lambda}_{41,4} = -v^{\lambda}_{43,1}, E_{45}v^{\lambda}_{41,i} = 0(i \in \overline{1,3})$ and $E_{34}v^{\lambda}_{41,3} = -v^{\lambda}_{47,1}, E_{34}v^{\lambda}_{41,i} = 0(i \in \{1, 2, 4\})$. Hence, $E_{45}E_{15}t_{15,45} = 0$ and $E_{34}E_{15}t_{15,45} = 0$ imply $v^{\lambda}_{41,3} = 0$ and $|\lambda|v^{\lambda}_{47,1} = 0$ in these two cases, respectively.

Case 13 $\quad wt(\xi_{0,3}) \in \{\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{46}, \lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{49}, \lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{50}, \lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{53}, \lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{54}, \lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{55}\}$

Observe $E_{25}v^{\lambda}_{26,3} = 2v^{\lambda}_{46,1}, E_{25}v^{\lambda}_{26,4} = -v^{\lambda}_{46,1}, E_{23}v^{\lambda}_{41,2} = -v^{\lambda}_{46,1}, E_{15}v^{\lambda}_{26,3} = -2v^{\lambda}_{49,1}, E_{15}v^{\lambda}_{26,4} = v^{\lambda}_{49,1}, E_{13}v^{\lambda}_{41,2} = v^{\lambda}_{49,1}, E_{12}v^{\lambda}_{46,1} = -v^{\lambda}_{49,1}, E_{25}v^{\lambda}_{41,2} = -v^{\lambda}_{54,1}, E_{35}v^{\lambda}_{46,1} = -v^{\lambda}_{54,1}$. Consider the equation

$$t_{35,45} = -v^{\lambda}_{26,3} - v^{\lambda}_{26,4} + E_{53}v^{\lambda}_{41,2} - E_{52}v^{\lambda}_{46,1} + E_{51}v^{\lambda}_{49,1}, \tag{5.65}$$

For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{46}, \overrightarrow{w}^{2\omega_1+\omega_4}_{50}$, the equation $E_{25}t_{35,45} = 0$ implies $2 + \lambda_2 + \lambda_3 + \lambda_4 = 0$. For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{49}, \overrightarrow{w}^{2\omega_1+\omega_4}_{53}$, the equation $E_{15}t_{35,45} = 0$ implies $3 + |\lambda| = 0$. For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{54}$, the equation $E_{35}E_{25}t_{35,45} = 0$ implies $(1 + \lambda_2)v^{\lambda}_{54,1} = 0$. For $wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{55}$, the equation $E_{35}E_{15}t_{35,45} = 0$ implies $\lambda_1 + \lambda_2 + 2 = 0$.

Case 14 $\quad wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{21}$

We have $v^{\lambda}_{21,3} + 2v^{\lambda}_{21,4} + E_{45}v^{\lambda}_{15,1} = 0$. And the equations $t_{25,45} = 0$ and $t_{24,45} = 0$ induce that $v^{\lambda}_{21,3} + v^{\lambda}_{21,4} = 0$, $v^{\lambda}_{15,1} = 0$. Thus, $v^{\lambda}_{21,3} = v^{\lambda}_{21,4} = 0$. Then

$$0 = E_{25}.t_{24,25} = E_{25}.[-v^{\lambda}_{4,1} + E_{53}v^{\lambda}_{17,1} - E_{52}(2v^{\lambda}_{21,1} + v^{\lambda}_{21,2})] = 0, \tag{5.66}$$

$$0 = E_{15}.t_{14,25} = E_{15}[-v^\lambda_{2,1} + E_{53}v^\lambda_{12,1} - E_{52}(v^\lambda_{18,1} + v^\lambda_{18,2} - v^\lambda_{18,3}) + E_{51}v^\lambda_{21,1}] = 0 \tag{5.67}$$

yield

$$2(1+\lambda_2+\lambda_3+\lambda_4)v^\lambda_{21,1} + (2+\lambda_2+\lambda_3+\lambda_4)v^\lambda_{21,2} = 0,\ (2+|\lambda|)v^\lambda_{21,1} = 0 \tag{5.68}$$

i.e. $v^\lambda_{21,i} = 0$ for $i \in \overline{1,4}$. A contradiction arises.

Case 15 $\quad wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{26}$

Note that

$$t_{24,35} = -v^\lambda_{8,1} + E_{53}\sum_{i=1}^{4} v^\lambda_{21,i} - E_{52}(v^\lambda_{26,2} + \sum_{i=1}^{4} v^\lambda_{26,i}). \tag{5.69}$$

We have $v^\lambda_{26,2} + v^\lambda_{26,3} + 2v^\lambda_{26,4} + E_{45}v^\lambda_{19,1} = 0$, $v^\lambda_{26,2} + v^\lambda_{26,3} + E_{23}v^\lambda_{21,3} = 0$ and $v^\lambda_{26,4} + E_{23}v^\lambda_{21,4} = 0$. And the equations $t_{35,45} = 0$, $t_{34,45} = 0$ and $t_{25,45} = 0$ induce that $v^\lambda_{26,3} + v^\lambda_{26,4} = 0$, $v^\lambda_{19,1} = 0$ and $v^\lambda_{26,2} + v^\lambda_{26,3} + v^\lambda_{26,4} = 0$. Thus, $v^\lambda_{26,i} = 0$ for $i \in \overline{2,4}$. The equation $t_{24,35} = 0$ induce that

$$E_{25}.[-v^\lambda_{8,1} + E_{53}(v^\lambda_{21,1} + v^\lambda_{21,2}) - E_{52}v^\lambda_{26,1}] = 0, \tag{5.70}$$

which means $\lambda_2+\lambda_3+\lambda_4 = 0$. A contradiction arises.

Case 16 $\quad wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{32}$

Note that $v^\lambda_{32,1} + v^\lambda_{32,2} + E_{14}v^\lambda_{18,3} = 0, v^\lambda_{32,1} + v^\lambda_{32,4} + E_{14}v^\lambda_{18,4} = 0, v^\lambda_{32,1} - 2v^\lambda_{32,3} - v^\lambda_{32,4} + E_{24}v^\lambda_{21,3} = 0, v^\lambda_{32,1} + v^\lambda_{32,2} + E_{24}v^\lambda_{21,4} = 0, 2v^\lambda_{32,1} + E_{34}v^\lambda_{26,3} = 0, v^\lambda_{32,1} + v^\lambda_{32,2} + E_{34}v^\lambda_{26,4} = 0$. And the equations follow:

$$t_{25,45} = v^\lambda_{21,3} + v^\lambda_{21,4} = 0, t_{35,45} = v^\lambda_{26,3} + v^\lambda_{26,4} = 0,$$

$$t_{34,45} = -v^\lambda_{19,1} + E_{53}v^\lambda_{32,3} = 0, t_{15,45} = -v^\lambda_{18,3} - v^\lambda_{18,4} + E_{53}v^\lambda_{30,1} = 0. \tag{5.71}$$

Thus,

$$-2v^\lambda_{32,1} + 2v^\lambda_{32,3} - v^\lambda_{32,2} + v^\lambda_{32,4} = 0,\ 3v^\lambda_{32,1} + v^\lambda_{32,2} = 0, v^\lambda_{32,2} + (\lambda_3+\lambda_4-1)v^\lambda_{32,3} = 0, 2v^\lambda_{32,1} + v^\lambda_{32,2} + v^\lambda_{32,4} = 0. \tag{5.72}$$

yield the contradiction:$2+\lambda_3+\lambda_4 = 0$.

Case 17 $\quad wt(\xi_{0,3}) = \lambda + \overrightarrow{w}^{2\omega_1+\omega_4}_{41}$

Observe that $-v^\lambda_{41,1} - v^\lambda_{41,2} + v^\lambda_{41,3} + v^\lambda_{41,4} + E_{24}v^\lambda_{28,1} = 0, v^\lambda_{41,2} + 2v^\lambda_{41,3} + v^\lambda_{41,4} + E_{34}v^\lambda_{34,1} = 0, -v^\lambda_{41,1} - v^\lambda_{41,2} + v^\lambda_{41,3} + E_{25}v^\lambda_{21,3} = 0, v^\lambda_{41,1} + v^\lambda_{41,2} + v^\lambda_{41,4} + E_{25}v^\lambda_{21,4} = 0, v^\lambda_{41,1} + 2v^\lambda_{41,2} + v^\lambda_{41,3} + E_{23}v^\lambda_{35,1} = 0, 2v^\lambda_{41,3} + E_{35}v^\lambda_{26,3} = 0, -v^\lambda_{41,2} - v^\lambda_{41,3} + v^\lambda_{41,4} + E_{35}v^\lambda_{26,4} = 0$. Then

$$0 = E_{25}t_{25,45} = E_{25}[-v^\lambda_{21,3} - v^\lambda_{21,4} + E_{53}v^\lambda_{35,1} - E_{52}(v^\lambda_{41,1} + v^\lambda_{41,2} + v^\lambda_{41,3})] = 0, \tag{5.73}$$

$$0 = E_{35}t_{35,45} = E_{35}[-v^\lambda_{26,3} - v^\lambda_{26,4} + E_{53}v^\lambda_{41,2}] = 0. \tag{5.74}$$

which mean

$$v^\lambda_{41,4} - v^\lambda_{41,1} - 2v^\lambda_{41,2} = (\sum_{i=2}^{4}\lambda_i)(\sum_{i=1}^{3} v^\lambda_{41,i}),\ v^\lambda_{41,3} - v^\lambda_{41,2} + v^\lambda_{41,4} + (\lambda_3+\lambda_4)v^\lambda_{41,2}). \tag{5.75}$$

We claim that one of $v^\lambda_{28,1}$ and $v^\lambda_{34,1}$ should be nonzero. Otherwise, the equation implies: $-v^\lambda_{41,1}-v^\lambda_{41,2}+v^\lambda_{41,3}+v^\lambda_{41,4}=0$ and $v^\lambda_{41,2}+2v^\lambda_{41,3}+v^\lambda_{41,4}=0$, which provide a contradiction. Now, $0=E_{15}t_{15,35}=E_{15}(E_{53}v^\lambda_{24,1}-E_{51}v^\lambda_{34,1})=E_{13}v^\lambda_{24,1}-\sum\limits_{i=1}^{4}h_iv^\lambda_{34,1}=0$ and $0=E_{15}t_{15,25}=E_{15}(E_{52}v^\lambda_{24,1}-E_{51}v^\lambda_{28,1})=E_{12}v^\lambda_{24,1}-\sum\limits_{i=1}^{4}h_iv^\lambda_{34,1}=0$. Thus $1+|\lambda|=0$. A contradiction arises.

Case 18 $\quad wt(\xi_{0,3})=\lambda+\overrightarrow{w}^{2\omega_1+\omega_4}_{18}$

Note that $E_{15}v^\lambda_{1,1}=v^\lambda_{18,1}-2v^\lambda_{18,3},E_{25}v^\lambda_{2,1}=-v^\lambda_{18,1}-v^\lambda_{18,2}-v^\lambda_{18,3},E_{13}v^\lambda_{7,1}=v^\lambda_{18,1}-2v^\lambda_{18,2}-2v^\lambda_{18,4},E_{13}v^\lambda_{3,1}=-2v^\lambda_{11,1},E_{23}v^\lambda_{12,1}=-v^\lambda_{18,1}-2v^\lambda_{18,2}-v^\lambda_{18,4},E_{34}v^\lambda_{11,1}=-v^\lambda_{18,2}-v^\lambda_{18,3}-2v^\lambda_{18,4}$. We have $v^\lambda_{11,1}=0$. Otherwise, $0=E_{15}t_{13,15}=-(2+|\lambda|)v^\lambda_{11,1}=0$. Hence,

$$0=-v^\lambda_{18,2}-v^\lambda_{18,3}-2v^\lambda_{18,4}=E_{34}v^\lambda_{11,1}. \tag{5.76}$$

Moreover,

$$0=t_{15,45}=v^\lambda_{18,3}+v^\lambda_{18,4},0=E_{15}t_{14,15}=3v^\lambda_{18,1}+2v^\lambda_{18,3}-2v^\lambda_{18,4}+|\lambda|(v^\lambda_{18,1}-v^\lambda_{18,4}). \tag{5.77}$$

Observe that

$$t_{14,25}=-v^\lambda_{2,1}+E_{53}v^\lambda_{12,1}-E_{52}(v^\lambda_{18,1}+v^\lambda_{18,2}-v^\lambda_{18,3})+E_{51}(v^\lambda_{21,1}-v^\lambda_{21,3}-v^\lambda_{21,4})=0. \tag{5.78}$$

The equation $E_{25}t_{14,25}=0$ induces

$$-v^\lambda_{18,2}+v^\lambda_{18,3}-v^\lambda_{18,4}-(\lambda_2+\lambda_3+\lambda_4)(v^\lambda_{18,1}+v^\lambda_{18,2}-v^\lambda_{18,3})=0. \tag{5.79}$$

All these equations yield the contradiction:

$$\frac{3+|\lambda|}{4+|\lambda|}=-(\lambda_2+\lambda_3+\lambda_4). \tag{5.80}$$

$\square$

5.4 Singular vectors of degree four

Theorem 5.6 All the possible degree four singular vectors are listed in the following:

$$d_{12}d_{13}d_{14}d_{15}v_\lambda,\ \lambda=(m,0,0,0),\ m\in\mathbb{N}.$$

Proof The leading term of any singular vector of degree four could be written as:

$$\xi_{0,4}=\sum_{j\in\overline{1,35}}v^{3\omega_1}_jv^\lambda_j, \tag{5.81}$$

which should satisfy $T_{5,123}.\xi_{0,4}=0$. Since

$$[T_{5,123}-(E_{53}\partial_{y_{12}}+E_{52}(-1)^{1+|13|}\partial_{y_{13}}+E_{51}(-1)^{|23|}\partial_{y_{23}}),(x_i\partial_{x_j})'_0]=0 \tag{5.82}$$

for $1\leq j<i\leq 5$, we have

$$[T_{5,123}-(E_{53}\partial_{y_{12}}+E_{52}(-1)^{1+|13|}\partial_{y_{13}}+E_{51}(-1)^{|23|}\partial_{y_{23}})].|_{V(3\omega_1)\otimes V(\lambda)}=0. \tag{5.83}$$

Hence,

$$T_{5,123}.\xi_{0,4} = [E_{53}\partial_{y_{12}} + E_{52}(-1)^{1+|13|}\partial_{y_{13}} + E_{51}(-1)^{|23|}\partial_{y_{23}}].\xi_{0,4} = 0. \tag{5.84}$$

Case 1 $\quad wt(\xi_{0,4}) = \overrightarrow{w}_1^{3\omega_1} = \mathrm{wt}(d_{12}d_{13}d_{14}d_{15}v_\lambda)$

The vector $d_{12}d_{13}d_{14}d_{15}v_\lambda$ is singular iff $\lambda = (m, 0, 0, 0)$.

Case 2 $\quad wt(\xi_{0,4}) \in \{\lambda + \overrightarrow{w}_2^{3\omega_1}, \lambda + \overrightarrow{w}_3^{3\omega_1}, \lambda + \overrightarrow{w}_5^{3\omega_1}, \lambda + \overrightarrow{w}_6^{3\omega_1}, \lambda + \overrightarrow{w}_8^{3\omega_1}, \lambda + \overrightarrow{w}_{10}^{3\omega_1}, \lambda + \overrightarrow{w}_{12}^{3\omega_1}, \lambda + \overrightarrow{w}_{13}^{3\omega_1}, \lambda + \overrightarrow{w}_{15}^{3\omega_1}, \lambda + \overrightarrow{w}_{16}^{3\omega_1}, \lambda + \overrightarrow{w}_{17}^{3\omega_1}, \lambda + \overrightarrow{w}_{22}^{3\omega_1}, \lambda + \overrightarrow{w}_{23}^{3\omega_1}, \lambda + \overrightarrow{w}_{27}^{3\omega_1}, \lambda + \overrightarrow{w}_{30}^{3\omega_1}\}$

In these cases, the equations are derived: $E_{51}v_i^\lambda - E_{52}v_j^\lambda = 0,\ E_{12}v_j^\lambda = -s_{ij}v_i^\lambda,\ \mathrm{wt}(v_i^\lambda) = \lambda, (i, j, s_{ij}) \in \{(2, 1, 3), (3, 2, 2), (5, 3, 1), (8, 6, 1), (10, 7, 2), (12, 9, 1), (13, 10, 1), (15, 11, 2), (16, 14, 1), (17, 15, 1)\}$. Then, the equation $E_{15}.(E_{51}v_i^\lambda - E_{52}v_j^\lambda) = (|\lambda| + s_{ij})v_i^\lambda = 0$ yields a contradiction.

Case 3 $\quad wt(\xi_{0,4}) \in \{\lambda + \overrightarrow{w}_4^{3\omega_1}, \lambda + \overrightarrow{w}_9^{3\omega_1}, \lambda + \overrightarrow{w}_{14}^{3\omega_1}, \lambda + \overrightarrow{w}_{19}^{3\omega_1}, \lambda + \overrightarrow{w}_{20}^{3\omega_1}, \lambda + \overrightarrow{w}_{21}^{3\omega_1}, \lambda + \overrightarrow{w}_{25}^{3\omega_1}, \lambda + \overrightarrow{w}_{26}^{3\omega_1}, \lambda + \overrightarrow{w}_{29}^{3\omega_1}, \lambda + \overrightarrow{w}_{33}^{3\omega_1}\}$

In these cases, we derive that: $E_{51}v_i^\lambda - E_{53}v_j^\lambda = 0,\ E_{13}v_j^\lambda = -t_{ij}v_i^\lambda,\ \mathrm{wt}(v_i^\lambda) = \lambda, (i, j, t_{ij}) \in \{(4, 1, 3), (9, 4, 2), (14, 7, 2), (19, 11, 2), (20, 9, 1), (21, 14, 1), (25, 18, 1), (26, 19, 1), (29, 24, 1), (33, 28, 1)\}$.Then, $E_{15}.(E_{51}v_i^\lambda - E_{53}v_j^\lambda) = (|\lambda| + t_{ij})v_i^\lambda = 0$ yields a contradiction.

Case 4 $\quad wt(\xi_{0,4}) \in \{\lambda + \overrightarrow{w}_7^{3\omega_1}, \lambda + \overrightarrow{w}_{11}^{3\omega_1}, \lambda + \overrightarrow{w}_{18}^{3\omega_1}, \lambda + \overrightarrow{w}_{24}^{3\omega_1}, \lambda + \overrightarrow{w}_{28}^{3\omega_1}\}$

In these cases, we have $\lambda_3 + \lambda_4 > 0$. And $E_{52}v_\lambda = E_{53}v_\lambda = 0$, i.e. $\lambda = (m, 0, 0, 0)$. A contradiction arises.

Case 5 $\quad wt(\xi_{0,4}) \in \{\lambda + \overrightarrow{w}_{31}^{3\omega_1}, \lambda + \overrightarrow{w}_{32}^{3\omega_1}, \lambda + \overrightarrow{w}_{34}^{3\omega_1}, \lambda + \overrightarrow{w}_{35}^{3\omega_1}\}$

Case 5.1 $\quad wt(\xi_{0,4}) \in \{\lambda + \overrightarrow{w}_{31}^{3\omega_1}, \lambda + \overrightarrow{w}_{32}^{3\omega_1}, \lambda + \overrightarrow{w}_{34}^{3\omega_1}\}$

In these cases, the following equations are derived: $E_{51}v_i^\lambda - E_{53}v_j^\lambda = 0,\ E_{13}v_j^\lambda = -q_{ij}v_i^\lambda,\ \mathrm{wt}(v_i^\lambda) = \lambda - \alpha_3, \mathrm{wt}(v_j^\lambda) = \lambda - \alpha_1 - \alpha_2 - \alpha_3, (i, j, t_{ij}) \in \{(25, 18, 1), (30, 24, 1), (34, 28, 1)$. Then, $E_{15}.(E_{51}v_i^\lambda - E_{53}v_j^\lambda) = (|\lambda| + q_{ij})v_i^\lambda = 0$ yields a contradiction.

Case 5.2 $\quad wt(\xi_{0,4}) = \lambda + \overrightarrow{w}_{35}^{3\omega_1}$

Note that $E_{51}v_{33}^\lambda - E_{53}v_{28}^\lambda = 0,\ E_{51}v_{30}^\lambda - E_{52}v_{28}^\lambda = 0,\ \mathrm{wt}(v_{33}^\lambda) = \lambda - \alpha_3 - \alpha_4,\ \mathrm{wt}(v_{30}^\lambda) = \lambda - \alpha_2 - \alpha_3 - 2\alpha_4,\ \mathrm{wt}(v_{28}^\lambda) = \lambda - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, E_{13}v_{28}^\lambda = -v_{33}^\lambda, E_{15}v_{28}^\lambda = -v_{35}^\lambda, E_{35}v_{33}^\lambda = -v_{35}^\lambda, E_{12}v_{28}^\lambda = -v_{30}^\lambda, E_{25}v_{30}^\lambda = -v_{35}^\lambda$. Then

$$E_{35}E_{15}(E_{51}v_{33}^\lambda - E_{53}v_{28}^\lambda) = 0 = E_{35}(|\lambda|v_{33}^\lambda + E_{53}v_{35}^\lambda) = -(\lambda_1 + \lambda_2)v_{35}^\lambda \tag{5.85}$$

induces $\lambda_1 = \lambda_2 = 0$. And

$$E_{35}E_{15}(E_{51}v_{30}^\lambda - E_{52}v_{28}^\lambda) = 0 = E_{35}[(|\lambda| - 1)v_{30}^\lambda + E_{52}v_{35}^\lambda] = (1 - \lambda_1)v_{35}^\lambda \tag{5.86}$$

induces $\lambda_1 = 1$. A contradiction arises. $\square$

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References

[EG] S. A. Edwards and M. D. Gould, A Projection based approach to the Clebsch-Gordan multiplicity problem for compact semisimple Lie groups: I. General formalism, *J. Phys. A.: Math. Gen.* 19 (1986), 1523-1529.

[FH] W. Fulton and J. Harris, Representation Theory: A First Course, volume **129** of Graduate Texts in Mathematics, Readings in Mathematics, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest, 1991.

[H] J. E. Humphreys, Introduction to Lie algebras and representation theory, Springer-Verlag, New York-Heidelberg-Berlin, 1972.

[K1] Kac, V., Classification of infinite-dimensional simple linearly compact Lie superalgebras, *Adv. Math.* 139, 1C55 (1998)

[K2] Kac, V., Classification of supersymmetries, *ICM 2002. Vol. I,* 319-344.

[KR1] Kac, V., Rudokov, A., Representaions of the exceptional Lie superalgebra $E(3,6)I$: Degeneracy conditions, *Transform. Groups 7* (2002), 67-86.

[KR2] Kac, V., Rudokov, A., Representaions of the exceptional Lie superalgebra $E(3,6)II$: Four series of degenerate modules, *Comm. Math. Phys.* 222 (2001), 611-661.

[KR3] Kac, V., Rudokov, A., Complexes of modules over the exceptional Lie superalgebra $E(3,8)$ and $E(5,10)$, *Int. Math. Res. Notices* 19 (2002), 1007-1025.

[LMNP] SP Li, R. V. Moody, M Nicolescu and J Patera, Verma bases for representations of classical simple Lie algebras, *J. Math. Phys.* 27 (1986), 668-677.

[MS] M. D. Gould and S. A. Edwards, Enveloping algebra annihilators and projection techniques for finite dimensional cyclic modules of a emidimple Lie algebra, *J. Math. Phys.* 25 (1984), 2848-2855.

[RS] K. N. Raghavana and P. Sankaranb, On Verma bases for representations of sl(n,$\mathbb{C}$), *J. Math. Phys.* 40, 2190 (1999),1999-2195.

[Sh] Shchepochkina, I., Exceptional simple infinite dimensional Lie superalgebras, *C. R. Acd. Bulg. Sci.* 36 (1983), 313-314.

[W] W. A. de Graaf, Lie Algebras: Theory and Algorithms, North-Holland Mathematical Library.

Table 1: **Weights and weight vectors for $\mathbf{sl}_5$ module** $V(\omega_2)$

i	$\overrightarrow{w}_i^{\omega_2}$	$v_i^{\omega_2}$	i	$\overrightarrow{w}_i^{\omega_2}$	$v_i^{\omega_2}$
1	$(0,1,0,0)=\omega_2$	d_{12}	6	$(0,-1,0,1)=\omega_2-\alpha_1-2\alpha_2-\alpha_3$	d_{34}
2	$(1,-1,1,0)=\omega_2-\alpha_2$	d_{13}	7	$(1,0,0,-1)=\omega_2-\alpha_2-\alpha_3-\alpha_4$	d_{15}
3	$(-1,0,1,0)=\omega_2-\alpha_1-\alpha_2$	d_{23}	8	$(-1,1,0,-1)=\omega_2-\alpha_1-\alpha_2-\alpha_3-\alpha_4$	d_{25}
4	$(1,0,-1,1)=\omega_2-\alpha_2-\alpha_3$	d_{14}	9	$(0,-1,1,-1)=\omega_2-\alpha_1-2\alpha_2-\alpha_3-\alpha_4$	d_{35}
5	$(-1,1,-1,1)=\omega_2-\alpha_1-\alpha_2-\alpha_3$	d_{24}	10	$(0,0,-1,0)=\omega_2-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	d_{45}

Table 2: **Decomposition for wedge module** $\Lambda^k W$ **(**$k\in\overline{1,10}$**)**

$\Lambda^k W$	irreducible components for $\Lambda^k W$	maximal vector for the irreducible components
$\wedge^1 W$	$V(\omega_2)$	d_{12}
$\wedge^2 W$	$V(\omega_1+\omega_3)$	$d_{12}\wedge d_{13}$
$\wedge^3 W$	$V(2\omega_3)\oplus V(2\omega_1+\omega_4)$	$d_{12}\wedge d_{13}\wedge d_{23}, d_{12}\wedge d_{13}\wedge d_{14}$
$\wedge^4 W$	$V(3\omega_1)\oplus V(\omega_1+\omega_3+\omega_4)$	$d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{15}, d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}$
$\wedge^5 W$	$V(2\omega_1+\omega_3)\oplus V(\omega_2+2\omega_4)$	$d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{15}, d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{24}$
$\wedge^6 W$	$V(3\omega_4)\oplus V(\omega_1+\omega_2+\omega_4)$	$d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{34}, d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{15}$
$\wedge^7 W$	$V(\omega_1+2\omega_4)\oplus V(2\omega_2)$	$d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{34}\wedge d_{15}, d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{15}\wedge d_{25}$
$\wedge^8 W$	$V(\omega_2+\omega_4)$	$d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{34}\wedge d_{15}\wedge d_{25}$
$\wedge^9 W$	$V(\omega_3)$	$d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{34}\wedge d_{15}\wedge d_{25}\wedge d_{35}$
$\wedge^{10} W$	$V(0)$	$d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{34}\wedge d_{15}\wedge d_{25}\wedge d_{35}\wedge d_{45}$

Table 3: **Tensor decomposition for** $V(k\omega_4)\otimes V(\mu)$

μ	highest weight in the decomposition $V(k\omega_4)\otimes V(\mu)$
ω_2	$\omega_{(0,1,0,k)}, \omega_{(1,0,0,k-1)}$
$\omega_1+\omega_3$	$\omega_{(1,0,1,k)}, \omega_{(0,0,1,k-1)}, \omega_{(1,1,0,k-1)}, \omega_{(0,1,0,k-2)}$
$2\omega_3$	$\omega_{(0,0,2,k)}, \omega_{(0,1,1,k-1)}, \omega_{(0,2,0,k-2)}$
$2\omega_1+\omega_4$	$\omega_{(2,0,0,k+1)}, \omega_{(1,0,0,k)}, \omega_{(0,0,0,k-1)}, \omega_{(2,0,1,k-1)}, \omega_{(1,0,1,k-2)}, \omega_{(0,0,1,k-3)}$
$\omega_1+\omega_3+\omega_4$	$\omega_{(1,0,1,k+1)}, \omega_{(0,0,1,k)}, \omega_{(1,1,0,k)}, \omega_{(0,1,0,k-1)}, \omega_{(1,0,2,k-1)}, \omega_{(0,0,2,k-2)}, \omega_{(1,1,1,k-2)}, \omega_{(0,1,1,k-3)}$
$3\omega_1$	$\omega_{(3,0,0,k)}, \omega_{(2,0,0,k-1)}, \omega_{(1,0,0,k-2)}, \omega_{(0,0,0,k-3)}$
$\omega_2+2\omega_4$	$\omega_{(0,1,0,k+2)}, \omega_{(1,0,0,k+1)}, \omega_{(0,1,1,k)}, \omega_{(1,0,1,k-1)}, \omega_{(0,1,2,k-2)}, \omega_{(1,0,2,k-3)}$
$2\omega_1+\omega_3$	$\omega_{(2,0,1,k)}, \omega_{(1,0,1,k-1)}, \omega_{(0,0,1,k-2)}, \omega_{(2,1,0,k-1)}, \omega_{(1,1,0,k-2)}, \omega_{(0,1,0,k-3)}$
$3\omega_4$	$\omega_{(0,0,0,k+3)}, \omega_{(0,0,1,k+1)}, \omega_{(0,0,2,k-1)}, \omega_{(0,0,3,k-3)}$
$\omega_1+\omega_2+\omega_4$	$\omega_{(1,1,0,k+1)}, \omega_{(0,1,0,k)}, \omega_{(2,0,0,k)}, \omega_{(1,0,0,k-1)}, \omega_{(1,1,1,k-1)}, \omega_{(0,1,1,k-2)}, \omega_{(2,0,1,k-2)}, \omega_{(1,0,1,k-3)}$
$\omega_1+2\omega_4$	$\omega_{(1,0,0,k+2)}, \omega_{(0,0,0,k+1)}, \omega_{(1,0,1,k)}, \omega_{(0,0,1,k-1)}, \omega_{(1,0,2,k-2)}, \omega_{(0,0,2,k-3)}$
$2\omega_2$	$\omega_{(0,2,0,k)}, \omega_{(1,1,0,k-1)}, \omega_{(2,0,0,k-2)}$
$\omega_2+\omega_4$	$\omega_{(0,1,0,k+1)}, \omega_{(1,0,0,k)}, \omega_{(0,1,1,k-1)}, \omega_{(1,0,1,k-2)},$
ω_3	$\omega_{(0,0,1,k)}, \omega_{(0,1,0,k-1)}$

Table 4: **The leading term l_μ**

μ	l_μ
ω_2	d_{12}, d_{15}
$\omega_1+\omega_3$	$d_{12}\wedge d_{13}, 2d_{12}\wedge d_{35}+d_{23}\wedge d_{15}-d_{13}\wedge d_{25}, d_{12}\wedge d_{15}, d_{15}\wedge d_{25}$
$2\omega_3$	$d_{12}\wedge d_{13}\wedge d_{23}, d_{12}\wedge d_{13}\wedge d_{25}-d_{12}\wedge d_{23}\wedge d_{15}, d_{12}\wedge d_{15}\wedge d_{25}$
$2\omega_1+\omega_4$	$d_{12}\wedge d_{13}\wedge d_{14}, d_{15}\wedge d_{25}\wedge d_{35}, 2d_{12}\wedge d_{13}\wedge d_{45}-2d_{12}\wedge d_{14}\wedge d_{35}+2d_{13}\wedge d_{14}\wedge d_{25}-d_{13}\wedge d_{24}\wedge d_{15}-d_{23}\wedge d_{14}\wedge d_{15}+d_{12}\wedge d_{34}\wedge d_{15}, d_{12}\wedge d_{35}\wedge d_{45}-d_{13}\wedge d_{25}\wedge d_{45}+d_{23}\wedge d_{15}\wedge d_{45}+d_{14}\wedge d_{25}\wedge d_{35}-d_{24}\wedge d_{15}\wedge d_{35}+d_{34}\wedge d_{15}\wedge d_{25}, d_{12}\wedge d_{13}\wedge d_{15}, d_{12}\wedge d_{15}\wedge d_{35}-d_{13}\wedge d_{15}\wedge d_{25}$
$\omega_1+\omega_3+\omega_4$	$d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}, d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{15}, 2d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{45}-d_{12}\wedge d_{13}\wedge d_{24}\wedge d_{35}+d_{12}\wedge d_{13}\wedge d_{34}\wedge d_{25}+d_{12}\wedge d_{23}\wedge d_{14}\wedge d_{35}-d_{12}\wedge d_{23}\wedge d_{34}\wedge d_{15}-d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{25}+d_{13}\wedge d_{23}\wedge d_{24}\wedge d_{15}, d_{12}\wedge d_{23}\wedge d_{14}\wedge d_{15}-2d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{25}+d_{12}\wedge d_{13}\wedge d_{24}\wedge d_{15}, d_{12}\wedge d_{13}\wedge d_{25}\wedge d_{45}-d_{12}\wedge d_{23}\wedge d_{15}\wedge d_{45}-d_{12}\wedge d_{14}\wedge d_{25}\wedge d_{35}+d_{12}\wedge d_{24}\wedge d_{15}\wedge d_{35}-d_{12}\wedge d_{34}\wedge d_{15}\wedge d_{25}, d_{12}\wedge d_{13}\wedge d_{25}\wedge d_{35}-d_{12}\wedge d_{23}\wedge d_{15}\wedge d_{35}+d_{13}\wedge d_{23}\wedge d_{15}\wedge d_{25}, d_{12}\wedge d_{13}\wedge d_{15}\wedge d_{25}, d_{12}\wedge d_{15}\wedge d_{25}\wedge d_{35}$
$3\omega_1$	$d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{15}, d_{12}\wedge d_{13}\wedge d_{15}\wedge d_{45}-d_{12}\wedge d_{14}\wedge d_{15}\wedge d_{35}+d_{13}\wedge d_{14}\wedge d_{15}\wedge d_{25}, d_{12}\wedge d_{15}\wedge d_{35}\wedge d_{45}-d_{13}\wedge d_{15}\wedge d_{25}\wedge d_{45}+d_{14}\wedge d_{15}\wedge d_{25}\wedge d_{35}, d_{15}\wedge d_{25}\wedge d_{35}\wedge d_{45}$
$\omega_2+2\omega_4$	$d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{24}, 2d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{45}+2d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{24}\wedge d_{35}-2d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{34}\wedge d_{25}+d_{12}\wedge d_{13}\wedge d_{24}\wedge d_{34}\wedge d_{15}+d_{12}\wedge d_{23}\wedge d_{14}\wedge d_{34}\wedge d_{15}-d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{15}, d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{25}-d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{24}\wedge d_{15}, 3d_{12}\wedge d_{23}\wedge d_{14}\wedge d_{15}\wedge d_{35}-3d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{15}\wedge d_{25}+4d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{15}\wedge d_{45}-2d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{25}\wedge d_{35}-d_{12}\wedge d_{13}\wedge d_{24}\wedge d_{15}\wedge d_{35}+d_{12}\wedge d_{13}\wedge d_{34}\wedge d_{15}\wedge d_{25}, d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{15}\wedge d_{25}, d_{12}\wedge d_{13}\wedge d_{15}\wedge d_{25}\wedge d_{35}$
$2\omega_1+\omega_3$	$d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{15}, 4d_{12}\wedge d_{13}\wedge d_{23}\wedge d_{15}\wedge d_{45}-2d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{25}\wedge d_{35}-d_{12}\wedge d_{13}\wedge d_{24}\wedge d_{15}\wedge d_{35}+3d_{12}\wedge d_{23}\wedge d_{14}\wedge d_{15}\wedge d_{35}-3d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{15}\wedge d_{25}+d_{12}\wedge d_{13}\wedge d_{34}\wedge d_{15}\wedge d_{25}, 2d_{12}\wedge d_{13}\wedge d_{25}\wedge d_{35}\wedge d_{45}-2d_{12}\wedge d_{23}\wedge d_{15}\wedge d_{35}\wedge d_{45}+d_{12}\wedge d_{34}\wedge d_{15}\wedge d_{25}\wedge d_{35}+2d_{13}\wedge d_{23}\wedge d_{15}\wedge d_{25}\wedge d_{45}-d_{13}\wedge d_{24}\wedge d_{15}\wedge d_{25}\wedge d_{35}+d_{23}\wedge d_{14}\wedge d_{15}\wedge d_{25}\wedge d_{35}, d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{15}\wedge d_{25}, d_{12}\wedge d_{15}\wedge d_{25}\wedge d_{35}\wedge d_{45}$
$3\omega_4$	$\hat{d}_{15}\wedge\hat{d}_{25}\wedge\hat{d}_{35}\wedge\hat{d}_{45}, \hat{d}_{34}\wedge\hat{d}_{15}\wedge\hat{d}_{25}\wedge\hat{d}_{45}-\hat{d}_{24}\wedge\hat{d}_{15}\wedge\hat{d}_{35}\wedge\hat{d}_{45}+\hat{d}_{14}\wedge\hat{d}_{25}\wedge\hat{d}_{35}\wedge\hat{d}_{45}, \hat{d}_{14}\wedge\hat{d}_{34}\wedge\hat{d}_{25}\wedge\hat{d}_{45}-\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{15}\wedge\hat{d}_{45}-\hat{d}_{14}\wedge\hat{d}_{24}\wedge\hat{d}_{35}\wedge\hat{d}_{45}, \hat{d}_{14}\wedge\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{45},$
$\omega_1+\omega_2+\omega_4$	$\hat{d}_{34}\wedge\hat{d}_{25}\wedge\hat{d}_{35}\wedge\hat{d}_{45}, \hat{d}_{12}\wedge\hat{d}_{34}\wedge\hat{d}_{35}\wedge\hat{d}_{45}-\hat{d}_{13}\wedge\hat{d}_{34}\wedge\hat{d}_{25}\wedge\hat{d}_{45}+\hat{d}_{23}\wedge\hat{d}_{34}\wedge\hat{d}_{15}\wedge\hat{d}_{45}+\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{15}\wedge\hat{d}_{35}-\hat{d}_{14}\wedge\hat{d}_{34}\wedge\hat{d}_{25}\wedge\hat{d}_{35}, \hat{d}_{23}\wedge\hat{d}_{24}\wedge\hat{d}_{35}\wedge\hat{d}_{45}-\hat{d}_{23}\wedge\hat{d}_{34}\wedge\hat{d}_{25}\wedge\hat{d}_{45}-\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{25}\wedge\hat{d}_{35}, \hat{d}_{12}\wedge\hat{d}_{23}\wedge\hat{d}_{34}\wedge\hat{d}_{45}+\hat{d}_{12}\wedge\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{35}-\hat{d}_{13}\wedge\hat{d}_{23}\wedge\hat{d}_{24}\wedge\hat{d}_{45}-\hat{d}_{13}\wedge\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{25}+\hat{d}_{23}\wedge\hat{d}_{14}\wedge\hat{d}_{24}\wedge\hat{d}_{35}-\hat{d}_{23}\wedge\hat{d}_{14}\wedge\hat{d}_{34}\wedge\hat{d}_{25}+2\hat{d}_{23}\wedge\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{15}, \hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{35}\wedge\hat{d}_{45}, \hat{d}_{13}\wedge\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{45}+2\hat{d}_{14}\wedge\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{35}-\hat{d}_{23}\wedge\hat{d}_{14}\wedge\hat{d}_{34}\wedge\hat{d}_{45}, \hat{d}_{23}\wedge\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{45}, \hat{d}_{23}\wedge\hat{d}_{14}\wedge\hat{d}_{24}\wedge\hat{d}_{34}$
$\omega_1+2\omega_4$	$\hat{d}_{25}\wedge\hat{d}_{35}\wedge\hat{d}_{45}, \hat{d}_{34}\wedge\hat{d}_{15}\wedge\hat{d}_{25}-\hat{d}_{24}\wedge\hat{d}_{15}\wedge\hat{d}_{35}+\hat{d}_{14}\wedge\hat{d}_{25}\wedge\hat{d}_{35}-\hat{d}_{23}\wedge\hat{d}_{15}\wedge\hat{d}_{45}+\hat{d}_{13}\wedge\hat{d}_{25}\wedge\hat{d}_{45}-\hat{d}_{12}\wedge\hat{d}_{35}\wedge\hat{d}_{45}, \hat{d}_{24}\wedge\hat{d}_{35}\wedge\hat{d}_{45}, 2\hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{15}-2\hat{d}_{14}\wedge\hat{d}_{34}\wedge\hat{d}_{25}-\hat{d}_{23}\wedge\hat{d}_{14}\wedge\hat{d}_{45}+\hat{d}_{13}\wedge\hat{d}_{24}\wedge\hat{d}_{45}-\hat{d}_{12}\wedge\hat{d}_{34}\wedge\hat{d}_{45}+2\hat{d}_{14}\wedge\hat{d}_{24}\wedge\hat{d}_{35}, \hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{45}, \hat{d}_{14}\wedge\hat{d}_{24}\wedge\hat{d}_{34},$
$2\omega_2$	$\hat{d}_{34}\wedge\hat{d}_{35}\wedge\hat{d}_{45}, \hat{d}_{24}\wedge\hat{d}_{34}\wedge\hat{d}_{35}+\hat{d}_{23}\wedge\hat{d}_{34}\wedge\hat{d}_{45}, \hat{d}_{23}\wedge\hat{d}_{24}\wedge\hat{d}_{34},$
$\omega_2+\omega_4$	$\hat{d}_{35}\wedge\hat{d}_{45}, \hat{d}_{23}\wedge\hat{d}_{45}+\hat{d}_{24}\wedge\hat{d}_{35}-\hat{d}_{34}\wedge\hat{d}_{25}, \hat{d}_{34}\wedge\hat{d}_{45}, \hat{d}_{24}\wedge\hat{d}_{34}$
ω_3	$\hat{d}_{45}, \hat{d}_{34}$

Table 5: **Weights and weight vectors for $\mathbf{sl}_5$ module $V(\omega_1+\omega_3)$**

(i,j)	$\overrightarrow{w}_i^{\omega_1+\omega_3}$	$v_{i,j}^{\omega_1+\omega_3}$	(i,j)	$\overrightarrow{w}_i^{\omega_1+\omega_3}$	$v_{i,j}^{\omega_1+\omega_3}$
(1,1)	$(1,0,1,0)=\omega_1+\omega_3$	$d_{12}\wedge d_{13}$	(18,1)	$(-1,-1,1,1)=\omega_1+\omega_3-2\alpha_1-2\alpha_2-\alpha_3$	$d_{23}\wedge d_{34}$
(2,1)	$(-1,1,1,0)=\omega_1+\omega_3-\alpha_1$	$d_{12}\wedge d_{23}$	(19,1)	$(-2,1,1,-1)=\omega_1+\omega_3-2\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{23}\wedge d_{25}$
(3,1)	$(1,1,-1,1)=\omega_1+\omega_3-\alpha_3$	$d_{12}\wedge d_{14}$	(20,1)	$(1,-1,-1,2)=\omega_1+\omega_3-\alpha_1-2\alpha_2-2\alpha_3$	$d_{14}\wedge d_{34}$
(4,1)	$(-1,2,-1,1)=\omega_1+\omega_3-\alpha_1-\alpha_3$	$d_{12}\wedge d_{24}$	(21,1)	$(1,-1,0,0)==\omega_1+\omega_3-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{14}\wedge d_{35}-d_{13}\wedge d_{45}$
(5,1)	$(1,1,0,-1)=\omega_1+\omega_3-\alpha_3-\alpha_4$	$d_{12}\wedge d_{15}$	(21,2)	$(1,-1,0,0)=\omega_1+\omega_3-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{14}\wedge d_{35}-d_{34}\wedge d_{15}$
(6,1)	$(0,-1,2,0)=\omega_1+\omega_3-\alpha_1-\alpha_2$	$d_{13}\wedge d_{23}$	(21,3)	$(1,-1,0,0)=\omega_1+\omega_3-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{34}\wedge d_{15}+d_{13}\wedge d_{45}$
(7,1)	$(2,-1,0,1)=\omega_1+\omega_3-\alpha_2-\alpha_3$	$d_{13}\wedge d_{14}$	(22,1)	$(-1,-1,2,-1)=\omega_1+\omega_3-2\alpha_1-2\alpha_2-\alpha_3-\alpha_4$	$d_{23}\wedge d_{35}$
(8,1)	$(0,0,0,1)=\omega_1+\omega_3-\alpha_1-\alpha_2-\alpha_3$	$d_{12}\wedge d_{34}-d_{13}\wedge d_{24}$	$(23,1)$	$(-1,0,-1,2)=\omega_1+\omega_3-2\alpha_1-2\alpha_2-2\alpha_3$	$d_{24}\wedge d_{34}$
(8,2)	$(0,0,0,1)=\omega_1+\omega_3-\alpha_1-\alpha_2-\alpha_3$	$d_{12}\wedge d_{34}+d_{23}\wedge d_{14}$	(24,1)	$(-2,2,-1,0)=\omega_1+\omega_3-2\alpha_1-\alpha_2-2\alpha_3-\alpha_4$	$d_{24}\wedge d_{25}$
(8,3)	$(0,0,0,1)=\omega_1+\omega_3-\alpha_1-\alpha_2-\alpha_3$	$d_{13}\wedge d_{24}-d_{23}\wedge d_{14}$	(25,1)	$(0,1,0,-2)=\omega_1+\omega_3-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4$	$d_{15}\wedge d_{25}$
(9,1)	$(-1,2,0,-1)=\omega_1+\omega_3-\alpha_1-\alpha_3-\alpha_4$	$d_{12}\wedge d_{25}$	(26,1)	$(-1,0,0,0)=\omega_1+\omega_3-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{34}\wedge d_{25}-d_{24}\wedge d_{35}$
(10,1)	$(2,-1,1,-1)=\omega_1+\omega_3-\alpha_2-\alpha_3-\alpha_4$	$d_{13}\wedge d_{15}$	(26,2)	$(-1,0,0,0)=\omega_1+\omega_3-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{23}\wedge d_{45}-d_{24}\wedge d_{35}$
(11,1)	$(0,0,1,-1)=\omega_1+\omega_3-\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{12}\wedge d_{35}-d_{13}\wedge d_{25}$	(26,3)	$(-1,0,0,0)=\omega_1+\omega_3-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{34}\wedge d_{25}+d_{23}\wedge d_{45}$
(11,2)	$(0,0,1,-1)=\omega_1+\omega_3-\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{12}\wedge d_{35}+d_{23}\wedge d_{15}$	(27,1)	$(1,0,-2,1)=\omega_1+\omega_3-\alpha_1-2\alpha_2-3\alpha_3-\alpha_4$	$d_{14}\wedge d_{45}$
(11,3)	$(0,0,1,-1)=\omega_1+\omega_3-\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{23}\wedge d_{15}-d_{13}\wedge d_{25}$	(28,1)	$(1,-1,1,-2)=\omega_1+\omega_3-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$	$d_{15}\wedge d_{35}$
(12,1)	$(1,-2,1,1)=\omega_1+\omega_3-\alpha_1-2\alpha_2-\alpha_3$	$d_{13}\wedge d_{34}$	(29,1)	$(-1,1,-2,1)=\omega_1+\omega_3-2\alpha_1-2\alpha_2-3\alpha_3-\alpha_4$	$d_{24}\wedge d_{45}$
(13,1)	$(-2,1,0,1)=\omega_1+\omega_3-2\alpha_1-\alpha_2-\alpha_3$	$d_{23}\wedge d_{24}$	(30,1)	$(0,-2,1,0)=\omega_1+\omega_3-2\alpha_1-3\alpha_2-2\alpha_3-\alpha_4$	$d_{34}\wedge d_{35}$
(14,1)	$(0,1,-2,2)=\omega_1+\omega_3-\alpha_1-\alpha_2-2\alpha_3$	$d_{14}\wedge d_{24}$	(31,1)	$(1,0,-1,-1)=\omega_1+\omega_3-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4$	$d_{15}\wedge d_{45}$
(15,1)	$(2,0,-1,0)=\omega_1+\omega_3-\alpha_2-2\alpha_3-\alpha_4$	$d_{14}\wedge d_{15}$	(32,1)	$(-1,0,1,-2)=\omega_1+\omega_3-2\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$	$d_{25}\wedge d_{35}$
(16,1)	$(0,1,-1,0)=\omega_1+\omega_3-\alpha_1-\alpha_2-2\alpha_3-\alpha_4$	$d_{12}\wedge d_{45}-d_{14}\wedge d_{25}$	(33,1)	$(0,-1,-1,1)=\omega_1+\omega_3-2\alpha_1-3\alpha_2-3\alpha_3-\alpha_4$	$d_{34}\wedge d_{45}$
(16,2)	$(0,1,-1,0)=\omega_1+\omega_3-\alpha_1-\alpha_2-2\alpha_3-\alpha_4$	$d_{12}\wedge d_{45}+d_{24}\wedge d_{15}$	(34,1)	$(-1,1,-1,-1)=\omega_1+\omega_3-2\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4$	$d_{25}\wedge d_{45}$
(16,3)	$(0,1,-1,0)=\omega_1+\omega_3-\alpha_1-\alpha_2-2\alpha_3-\alpha_4$	$d_{14}\wedge d_{25}-d_{24}\wedge d_{15}$	(35,1)	$(0,-1,0,-1)=\omega_1+\omega_3-2\alpha_1-3\alpha_2-3\alpha_3-2\alpha_4$	$d_{35}\wedge d_{45}$
(17,1)	$(1,-2,2,-1)=\omega_1+\omega_3-\alpha_1-2\alpha_2-\alpha_3-\alpha_4$	$d_{13}\wedge d_{35}$			

Table 6: **Weights and weight vectors for $\mathbf{sl}_5$ module** $V(2\omega_1+\omega_4)$

(i,j)	$\overrightarrow{w}_i^{2\omega_1+\omega_4}$	$v_{i,j}^{2\omega_1+\omega_4}$
(1,1)	$(2,0,0,1)=2\omega_1+\omega_4$	$d_{12}\wedge d_{13}\wedge d_{14}$
(2,1)	$(0,1,0,1)=2\omega_1+\omega_4-\alpha_1$	$d_{12}\wedge d_{23}\wedge d_{14}+d_{12}\wedge d_{13}\wedge d_{24}$
(3,1)	$(2,0,1,-1)=2\omega_1+\omega_4-\alpha_4$	$d_{12}\wedge d_{13}\wedge d_{15}$
(4,1)	$(-2,2,0,1)=2\omega_1+\omega_4-2\alpha_1$	$d_{12}\wedge d_{23}\wedge d_{24}$
(5,1)	$(1,-1,1,1)=2\omega_1+\omega_4-\alpha_1-\alpha_2$	$d_{13}\wedge d_{23}\wedge d_{14}+d_{12}\wedge d_{13}\wedge d_{34}$
(6,1)	$(0,1,1,-1)=2\omega_1+\omega_4-\alpha_1-\alpha_4$	$d_{12}\wedge d_{23}\wedge d_{15}+d_{12}\wedge d_{13}\wedge d_{25}$
(7,1)	$(2,1,-1,0)=2\omega_1+\omega_4-\alpha_3-\alpha_4$	$d_{12}\wedge d_{14}\wedge d_{15}$
(8,1)	$(-1,0,1,1)=2\omega_1+\omega_4-2\alpha_1-\alpha_2$	$d_{13}\wedge d_{23}\wedge d_{24}+d_{12}\wedge d_{23}\wedge d_{34}$
(9,1)	$(-2,2,1,-1)=2\omega_1+\omega_4-2\alpha_1-\alpha_4$	$d_{12}\wedge d_{23}\wedge d_{25}$
(10,1)	$(1,0,-1,2)=2\omega_1+\omega_4-\alpha_1-\alpha_2-\alpha_3$	$-d_{13}\wedge d_{14}\wedge d_{24}+d_{12}\wedge d_{14}\wedge d_{34}$
(11,1)	$(1,-1,2,-1)=2\omega_1+\omega_4-\alpha_1-\alpha_2-\alpha_4$	$d_{13}\wedge d_{23}\wedge d_{15}+d_{12}\wedge d_{13}\wedge d_{35}$
(12,1)	$(0,2,-1,0)=2\omega_1+\omega_4-\alpha_1-\alpha_3-\alpha_4$	$d_{12}\wedge d_{24}\wedge d_{15}+d_{12}\wedge d_{14}\wedge d_{25}$
(13,1)	$(3,-1,0,0)=2\omega_1+\omega_4-\alpha_2-\alpha_3-\alpha_4$	$d_{13}\wedge d_{14}\wedge d_{15}$
(14,1)	$(0,-2,2,1)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2$	$d_{13}\wedge d_{23}\wedge d_{34}$
(15,1)	$(-1,1,-1,2)=2\omega_1+\omega_4-2\alpha_1-\alpha_2-\alpha_3$	$d_{12}\wedge d_{24}\wedge d_{34}-d_{23}\wedge d_{14}\wedge d_{24}$
(16,1)	$(-1,0,2,-1)=2\omega_1+\omega_4-2\alpha_1-\alpha_2-\alpha_4$	$d_{12}\wedge d_{23}\wedge d_{35}+d_{13}\wedge d_{23}\wedge d_{25}$
(17,1)	$(-2,3,-1,0)=2\omega_1+\omega_4-2\alpha_1-\alpha_3-\alpha_4$	$d_{12}\wedge d_{24}\wedge d_{25}$
(18,1)	$(1,0,0,0)=2\omega_1+\omega_4-\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{13}\wedge d_{24}\wedge d_{15}+d_{13}\wedge d_{14}\wedge d_{25}+d_{23}\wedge d_{14}\wedge d_{15},$
(18,2)	$(1,0,0,0)=2\omega_1+\omega_4-\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{13}\wedge d_{24}\wedge d_{15}+d_{13}\wedge d_{14}\wedge d_{25}+d_{12}\wedge d_{14}\wedge d_{35}+d_{12}\wedge d_{34}\wedge d_{15},$
(18,3)	$(1,0,0,0)=2\omega_1+\omega_4-\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{13}\wedge d_{24}\wedge d_{15}-d_{13}\wedge d_{14}\wedge d_{25}-d_{12}\wedge d_{34}\wedge d_{15}+d_{12}\wedge d_{14}\wedge d_{35}$
(18,4)	$(1,0,0,0)=2\omega_1+\omega_4-\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$-d_{23}\wedge d_{14}\wedge d_{15}+d_{13}\wedge d_{24}\wedge d_{15}+d_{12}\wedge d_{14}\wedge d_{35}+d_{12}\wedge d_{13}\wedge d_{45}$
(19,1)	$(0,-1,0,2)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-\alpha_3$	$d_{13}\wedge d_{24}\wedge d_{34}-d_{23}\wedge d_{14}\wedge d_{34}$
(20,1)	$(0,-2,3,-1)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-\alpha_4$	$d_{13}\wedge d_{23}\wedge d_{35}$
(21,1)	$(-1,1,0,0)=2\omega_1+\omega_4-2\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{12}\wedge d_{34}\wedge d_{25}+d_{12}\wedge d_{24}\wedge d_{35}+2d_{13}\wedge d_{24}\wedge d_{25}+d_{23}\wedge d_{14}\wedge d_{25}+d_{23}\wedge d_{24}\wedge d_{15}$
(21,2)	$(-1,1,0,0)=2\omega_1+\omega_4-2\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{12}\wedge d_{24}\wedge d_{35}+d_{12}\wedge d_{34}\wedge d_{25}+d_{13}\wedge d_{24}\wedge d_{25}$
(21,3)	$(-1,1,0,0)=2\omega_1+\omega_4-2\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$-d_{23}\wedge d_{14}\wedge d_{25}+d_{13}\wedge d_{24}\wedge d_{25}+d_{12}\wedge d_{24}\wedge d_{35}+d_{12}\wedge d_{23}\wedge d_{45}$
(21,4)	$(-1,1,0,0)=2\omega_1+\omega_4-2\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{12}\wedge d_{24}\wedge d_{35}-d_{12}\wedge d_{34}\wedge d_{25}+d_{23}\wedge d_{24}\wedge d_{15}-d_{23}\wedge d_{14}\wedge d_{25}$
(22,1)	$(2,-2,1,0)=2\omega_1+\omega_4-\alpha_1-2\alpha_2-\alpha_3-\alpha_4$	$d_{13}\wedge d_{34}\wedge d_{15}+d_{13}\wedge d_{14}\wedge d_{35}$
(23,1)	$(1,1,-2,1)=2\omega_1+\omega_4-\alpha_1-\alpha_2-2\alpha_3-\alpha_4$	$d_{12}\wedge d_{14}\wedge d_{45}+d_{14}\wedge d_{24}\wedge d_{15}$
(24,1)	$(1,0,1,-2)=2\omega_1+\omega_4-\alpha_1-\alpha_2-\alpha_3-2\alpha_4$	$d_{12}\wedge d_{15}\wedge d_{35}-d_{13}\wedge d_{15}\wedge d_{25}$
(25,1)	$(0,0,-2,3)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-2\alpha_3$	$d_{14}\wedge d_{24}\wedge d_{34}$
(26,1)	$(0,-1,1,0)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-\alpha_3-\alpha_4$	$d_{13}\wedge d_{24}\wedge d_{35}+d_{13}\wedge d_{34}\wedge d_{25}+d_{23}\wedge d_{14}\wedge d_{35}+d_{23}\wedge d_{34}\wedge d_{15}$
(26,2)	$(0,-1,1,0)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-\alpha_3-\alpha_4$	$-d_{23}\wedge d_{14}\wedge d_{35}+d_{13}\wedge d_{34}\wedge d_{25}+2d_{13}\wedge d_{24}\wedge d_{35}+d_{12}\wedge d_{34}\wedge d_{35}+d_{13}\wedge d_{23}\wedge d_{45}$
(26,3)	$(0,-1,1,0)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-\alpha_3-\alpha_4$	$d_{13}\wedge d_{24}\wedge d_{35}+d_{13}\wedge d_{23}\wedge d_{45}-d_{23}\wedge d_{14}\wedge d_{35}$
(26,4)	$(0,-1,1,0)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-\alpha_3-\alpha_4$	$d_{23}\wedge d_{34}\wedge d_{15}-d_{23}\wedge d_{14}\wedge d_{35}-d_{13}\wedge d_{34}\wedge d_{25}+d_{13}\wedge d_{24}\wedge d_{35}$
(27,1)	$(-1,2,-2,1)=2\omega_1+\omega_4-2\alpha_1-\alpha_2-2\alpha_3-\alpha_4$	$d_{14}\wedge d_{24}\wedge d_{25}+d_{12}\wedge d_{24}\wedge d_{45}$
(28,1)	$(-1,1,1,-2)=2\omega_1+\omega_4-2\alpha_1-\alpha_2-\alpha_3-2\alpha_4$	$d_{12}\wedge d_{25}\wedge d_{35}-d_{23}\wedge d_{15}\wedge d_{25}$
(29,1)	$(2,-1,-1,1)=2\omega_1+\omega_4-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{14}\wedge d_{34}\wedge d_{15}+d_{13}\wedge d_{14}\wedge d_{45}$
(30,1)	$(1,1,-1,-1)=2\omega_1+\omega_4-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4$	$d_{12}\wedge d_{15}\wedge d_{45}-d_{14}\wedge d_{15}\wedge d_{25}$

Table 7: **Weights and weight vectors for $\mathbf{sl}_5$ module** $V(2\omega_1+\omega_4)$

(i,j)	$\overrightarrow{w}_i^{2\omega_1+\omega_4}$	$v_{i,j}^{2\omega_1+\omega_4}$
(31,1)	$(-3,2,0,0)=2\omega_1+\omega_4-3\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$d_{23}\wedge d_{24}\wedge d_{25}$
(32,1)	$(0,0,-1,1)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{13}\wedge d_{24}\wedge d_{45}+2d_{14}\wedge d_{24}\wedge d_{35}-d_{14}\wedge d_{34}\wedge d_{25}-d_{23}\wedge d_{14}\wedge d_{45}+d_{24}\wedge d_{34}\wedge d_{15}$
(32,2)	$(0,0,-1,1)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{14}\wedge d_{24}\wedge d_{35}-d_{14}\wedge d_{34}\wedge d_{25}+d_{24}\wedge d_{34}\wedge d_{15}$
(32,3)	$(0,0,-1,1)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{12}\wedge d_{34}\wedge d_{45}+d_{13}\wedge d_{24}\wedge d_{45}+d_{14}\wedge d_{34}\wedge d_{25}+d_{14}\wedge d_{24}\wedge d_{35}$
(32,4)	$(0,0,-1,1)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{13}\wedge d_{24}\wedge d_{45}+d_{23}\wedge d_{14}\wedge d_{45}+d_{14}\wedge d_{34}\wedge d_{25}+d_{24}\wedge d_{34}\wedge d_{15}$
(33,1)	$(-2,0,1,0)=2\omega_1+\omega_4-3\alpha_1-2\alpha_2-\alpha_3-\alpha_4$	$d_{23}\wedge d_{24}\wedge d_{35}+d_{23}\wedge d_{34}\wedge d_{25}$
(34,1)	$(0,-1,2,-2)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-\alpha_3-2\alpha_4$	$d_{13}\wedge d_{25}\wedge d_{35}-d_{23}\wedge d_{15}\wedge d_{35}$
(35,1)	$(-1,2,-1,-1)=2\omega_1+\omega_4-2\alpha_1-\alpha_2-2\alpha_3-2\alpha_4$	$d_{12}\wedge d_{25}\wedge d_{45}-d_{24}\wedge d_{15}\wedge d_{25}$
(36,1)	$(2,-1,0,-1)=2\omega_1+\omega_4-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$	$d_{13}\wedge d_{15}\wedge d_{45}-d_{14}\wedge d_{15}\wedge d_{35}$
(37,1)	$(1,-3,2,0)=2\omega_1+\omega_4-2\alpha_1-3\alpha_2-\alpha_3-\alpha_4$	$d_{13}\wedge d_{34}\wedge d_{35}$
(38,1)	$(-1,-2,2,0)=2\omega_1+\omega_4-3\alpha_1-3\alpha_2-\alpha_3-\alpha_4$	$d_{23}\wedge d_{34}\wedge d_{35}$
(39,1)	$(-2,1,-1,1)=2\omega_1+\omega_4-3\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$	$d_{23}\wedge d_{24}\wedge d_{45}+d_{24}\wedge d_{34}\wedge d_{25}$
(40,1)	$(1,-2,0,1)=2\omega_1+\omega_4-2\alpha_1-3\alpha_2-2\alpha_3-\alpha_4$	$d_{13}\wedge d_{34}\wedge d_{45}+d_{14}\wedge d_{34}\wedge d_{35}$
(41,1)	$(0,0,0,-1)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$	$d_{13}\wedge d_{25}\wedge d_{45}+d_{23}\wedge d_{15}\wedge d_{45}-d_{14}\wedge d_{25}\wedge d_{35}-d_{24}\wedge d_{15}\wedge d_{35}$
(41,2)	$(0,0,0,-1)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$	$d_{12}\wedge d_{35}\wedge d_{45}+d_{13}\wedge d_{25}\wedge d_{45}-d_{24}\wedge d_{15}\wedge d_{35}-d_{34}\wedge d_{15}\wedge d_{25}$
(41,3)	$(0,0,0,-1)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$	$d_{13}\wedge d_{25}\wedge d_{45}-d_{23}\wedge d_{15}\wedge d_{45}+d_{14}\wedge d_{25}\wedge d_{35}-d_{24}\wedge d_{15}\wedge d_{35}$
(41,4)	$(0,0,0,-1)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$	$d_{14}\wedge d_{25}\wedge d_{35}-d_{24}\wedge d_{15}\wedge d_{35}+d_{34}\wedge d_{15}\wedge d_{25}$
(42,1)	$(0,1,-3,2)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-3\alpha_3-\alpha_4$	$d_{14}\wedge d_{24}\wedge d_{45}$
(43,1)	$(0,0,1,-3)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-2\alpha_3-3\alpha_4$	$d_{15}\wedge d_{25}\wedge d_{35}$
(44,1)	$(-1,-1,0,1)=2\omega_1+\omega_4-3\alpha_1-3\alpha_2-2\alpha_3-\alpha_4$	$d_{23}\wedge d_{34}\wedge d_{45}+d_{24}\wedge d_{34}\wedge d_{35}$
(45,1)	$(-2,1,0,-1)=2\omega_1+\omega_4-3\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$	$d_{23}\wedge d_{25}\wedge d_{45}-d_{24}\wedge d_{25}\wedge d_{35}$
(46,1)	$(1,-2,1,-1)=2\omega_1+\omega_4-2\alpha_1-3\alpha_2-2\alpha_3-2\alpha_4$	$d_{13}\wedge d_{35}\wedge d_{45}-d_{34}\wedge d_{15}\wedge d_{35}$
(47,1)	$(0,1,-2,0)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4$	$d_{14}\wedge d_{25}\wedge d_{45}-d_{24}\wedge d_{15}\wedge d_{45}$
(48,1)	$(1,-1,-2,2)=2\omega_1+\omega_4-2\alpha_1-3\alpha_2-3\alpha_3-\alpha_4$	$d_{14}\wedge d_{34}\wedge d_{45}$
(49,1)	$(-1,-1,1,-1)=2\omega_1+\omega_4-3\alpha_1-3\alpha_2-2\alpha_3-2\alpha_4$	$d_{23}\wedge d_{35}\wedge d_{45}-d_{34}\wedge d_{25}\wedge d_{35}$
(50,1)	$(1,-1,-1,0)=2\omega_1+\omega_4-2\alpha_1-3\alpha_2-3\alpha_3-2\alpha_4$	$d_{14}\wedge d_{35}\wedge d_{45}-d_{34}\wedge d_{15}\wedge d_{45}$
(51,1)	$(-1,0,-2,2)=2\omega_1+\omega_4-3\alpha_1-3\alpha_2-3\alpha_3-\alpha_4$	$d_{24}\wedge d_{34}\wedge d_{45}$
(52,1)	$(0,1,-1,-2)=2\omega_1+\omega_4-2\alpha_1-2\alpha_2-3\alpha_3-3\alpha_4$	$d_{15}\wedge d_{25}\wedge d_{45}$
(53,1)	$(-1,0,-1,0)=2\omega_1+\omega_4-3\alpha_1-3\alpha_2-3\alpha_3-2\alpha_4$	$d_{24}\wedge d_{35}\wedge d_{45}-d_{34}\wedge d_{25}\wedge d_{45}$
(54,1)	$(1,-1,0,-2)=2\omega_1+\omega_4-2\alpha_1-3\alpha_2-3\alpha_3-3\alpha_4$	$d_{15}\wedge d_{35}\wedge d_{45}$
(55,1)	$(-1,0,0,-2)=2\omega_1+\omega_4-3\alpha_1-3\alpha_2-3\alpha_3-3\alpha_4$	$d_{25}\wedge d_{35}\wedge d_{45}$

Table 8: **Weights and weight vectors for $\mathbf{sl}_5$ module** $V(3\omega_1)$

i	$\overrightarrow{w}_i^{3\omega_1}$	$v_i^{3\omega_1}$
1	$(3,0,0,0) = 3\omega_1$	$d_{12} \wedge d_{13} \wedge d_{14} \wedge d_{15}$
2	$(1,1,0,0) = 3\omega_1 - \alpha_1$	$d_{12}\wedge d_{23}\wedge d_{14}\wedge d_{15}+d_{12}\wedge d_{13}\wedge d_{24}\wedge d_{15}+d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{25}$
3	$(-1,2,0,0) = 3\omega_1 - 2\alpha_1$	$d_{12}\wedge d_{23}\wedge d_{24}\wedge d_{15}+d_{12}\wedge d_{23}\wedge d_{14}\wedge d_{25}+d_{12}\wedge d_{13}\wedge d_{24}\wedge d_{25}$
4	$(2,-1,1,0) = 3\omega_1 - \alpha_1 - \alpha_2$	$d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{15}+d_{12}\wedge d_{13}\wedge d_{34}\wedge d_{15}+d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{35}$
5	$(-3,3,0,0) = 3\omega_1 - 3\alpha_1$	$d_{12} \wedge d_{23} \wedge d_{24} \wedge d_{25}$
6	$(0,0,1,0) = 3\omega_1 - 2\alpha_1 - \alpha_2$	$d_{13}\wedge d_{23}\wedge d_{24}\wedge d_{15}+d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{25}+d_{12}\wedge d_{23}\wedge d_{34}\wedge d_{15}+$ $d_{12}\wedge d_{13}\wedge d_{34}\wedge d_{25}+d_{12}\wedge d_{23}\wedge d_{14}\wedge d_{35}+d_{12}\wedge d_{13}\wedge d_{24}\wedge d_{35}$
7	$(2,0,-1,1) = 3\omega_1 - \alpha_1 - \alpha_2 - \alpha_3$	$-d_{13}\wedge d_{14}\wedge d_{24}\wedge d_{15}+d_{12}\wedge d_{13}\wedge d_{14}\wedge d_{45}+d_{12}\wedge d_{14}\wedge d_{34}\wedge d_{15}$
8	$(-2,1,1,0) = 3\omega_1 - 3\alpha_1 - \alpha_2$	$d_{13}\wedge d_{23}\wedge d_{24}\wedge d_{25}+d_{12}\wedge d_{23}\wedge d_{34}\wedge d_{25}+d_{12}\wedge d_{23}\wedge d_{24}\wedge d_{35}$
9	$(1,-2,2,0) = 3\omega_1 - 2\alpha_1 - 2\alpha_2$	$d_{13}\wedge d_{23}\wedge d_{34}\wedge d_{15}+d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{35}+d_{12}\wedge d_{13}\wedge d_{34}\wedge d_{35}$
10	$(0,1,-1,1) = 3\omega_1 - 2\alpha_1 - \alpha_2 - \alpha_3$	$-d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{15}-d_{13}\wedge d_{14}\wedge d_{24}\wedge d_{25}+d_{12}\wedge d_{23}\wedge d_{14}\wedge d_{45}+$ $d_{12}\wedge d_{13}\wedge d_{24}\wedge d_{45}+d_{12}\wedge d_{24}\wedge d_{34}\wedge d_{15}+d_{12}\wedge d_{14}\wedge d_{34}\wedge d_{25}$
11	$(2,0,0,-1) = 3\omega_1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$	$d_{13}\wedge d_{14}\wedge d_{15}\wedge d_{25}+d_{12}\wedge d_{13}\wedge d_{15}\wedge d_{45}-d_{12}\wedge d_{14}\wedge d_{15}\wedge d_{35}$
12	$(-1,-1,2,0) = 3\omega_1 - 3\alpha_1 - 2\alpha_2$	$d_{13}\wedge d_{23}\wedge d_{34}\wedge d_{25}+d_{13}\wedge d_{23}\wedge d_{24}\wedge d_{35}+d_{12}\wedge d_{23}\wedge d_{34}\wedge d_{35}$
13	$(-2,2,-1,1) = 3\omega_1 - 3\alpha_1 - \alpha_2 - \alpha_3$	$-d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{25}+d_{12}\wedge d_{24}\wedge d_{34}\wedge d_{25}+d_{12}\wedge d_{23}\wedge d_{24}\wedge d_{45}$
14	$(1,-1,0,1) = 3\omega_1 - 2\alpha_1 - 2\alpha_2 - \alpha_3$	$-d_{23}\wedge d_{14}\wedge d_{34}\wedge d_{15}-d_{13}\wedge d_{14}\wedge d_{24}\wedge d_{35}+d_{13}\wedge d_{23}\wedge d_{14}\wedge d_{45}+$ $d_{12}\wedge d_{13}\wedge d_{34}\wedge d_{45}+d_{13}\wedge d_{24}\wedge d_{34}\wedge d_{15}+d_{12}\wedge d_{14}\wedge d_{34}\wedge d_{35}$
15	$(0,1,0,-1) = 3\omega_1 - 2\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$	$d_{23}\wedge d_{14}\wedge d_{15}\wedge d_{25}+d_{13}\wedge d_{24}\wedge d_{15}\wedge d_{25}+d_{12}\wedge d_{23}\wedge d_{15}\wedge d_{45}+$ $d_{12}\wedge d_{13}\wedge d_{25}\wedge d_{45}-d_{12}\wedge d_{24}\wedge d_{15}\wedge d_{35}-d_{12}\wedge d_{14}\wedge d_{25}\wedge d_{35}$
16	$(-1,0,0,1) = 3\omega_1 - 3\alpha_1 - 2\alpha_2 - \alpha_3$	$-d_{23}\wedge d_{14}\wedge d_{34}\wedge d_{25}-d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{35}+d_{13}\wedge d_{24}\wedge d_{34}\wedge d_{25}+$ $d_{13}\wedge d_{23}\wedge d_{24}\wedge d_{45}+d_{12}\wedge d_{24}\wedge d_{34}\wedge d_{35}+d_{12}\wedge d_{23}\wedge d_{34}\wedge d_{45}$
17	$(-2,2,0,-1) = 3\omega_1 - 3\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$	$d_{23}\wedge d_{24}\wedge d_{15}\wedge d_{25}-d_{12}\wedge d_{24}\wedge d_{25}\wedge d_{35}+d_{12}\wedge d_{23}\wedge d_{25}\wedge d_{45}$
18	$(1,0,-2,2) = 3\omega_1 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3$	$d_{14}\wedge d_{24}\wedge d_{34}\wedge d_{15}-d_{13}\wedge d_{14}\wedge d_{24}\wedge d_{45}+d_{12}\wedge d_{14}\wedge d_{34}\wedge d_{45}$
19	$(1,-1,1,-1) = 3\omega_1 - 2\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4$	$d_{23}\wedge d_{14}\wedge d_{15}\wedge d_{35}+d_{13}\wedge d_{34}\wedge d_{15}\wedge d_{25}+d_{13}\wedge d_{23}\wedge d_{15}\wedge d_{45}+$ $d_{12}\wedge d_{13}\wedge d_{35}\wedge d_{45}-d_{12}\wedge d_{34}\wedge d_{15}\wedge d_{35}-d_{13}\wedge d_{14}\wedge d_{25}\wedge d_{35}$
20	$(0,-3,3,0) = 3\omega_1 - 3\alpha_1 - 3\alpha_2$	$d_{13} \wedge d_{23} \wedge d_{34} \wedge d_{35}$
21	$(0,-2,1,1) = 3\omega_1 - 3\alpha_1 - 3\alpha_2 - \alpha_3$	$-d_{23}\wedge d_{14}\wedge d_{34}\wedge d_{35}+d_{13}\wedge d_{23}\wedge d_{34}\wedge d_{45}+d_{13}\wedge d_{24}\wedge d_{34}\wedge d_{35}$
22	$(-1,1,-2,2) = 3\omega_1 - 3\alpha_1 - 2\alpha_2 - 2\alpha_3$	$d_{14}\wedge d_{24}\wedge d_{34}\wedge d_{25}-d_{23}\wedge d_{14}\wedge d_{24}\wedge d_{45}+d_{12}\wedge d_{24}\wedge d_{34}\wedge d_{45}$
23	$(1,0,-1,0) = 3\omega_1 - 3\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4$	$d_{23}\wedge d_{24}\wedge d_{15}\wedge d_{35}+d_{23}\wedge d_{34}\wedge d_{15}\wedge d_{25}+d_{13}\wedge d_{23}\wedge d_{25}\wedge d_{45}+$ $d_{12}\wedge d_{23}\wedge d_{35}\wedge d_{45}-d_{12}\wedge d_{34}\wedge d_{25}\wedge d_{35}-d_{13}\wedge d_{24}\wedge d_{25}\wedge d_{35}$
24	$(0,-1,-1,2) = 3\omega_1 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$	$-d_{14}\wedge d_{24}\wedge d_{15}\wedge d_{35}+d_{14}\wedge d_{34}\wedge d_{15}\wedge d_{25}+d_{13}\wedge d_{24}\wedge d_{15}\wedge d_{45}+$ $d_{12}\wedge d_{14}\wedge d_{35}\wedge d_{45}-d_{12}\wedge d_{34}\wedge d_{15}\wedge d_{45}-d_{13}\wedge d_{14}\wedge d_{25}\wedge d_{45}$
25	$(0,-2,2,-1) = 3\omega_1 - 3\alpha_1 - 3\alpha_2 - 2\alpha_3$	$d_{14}\wedge d_{24}\wedge d_{34}\wedge d_{35}-d_{23}\wedge d_{14}\wedge d_{34}\wedge d_{45}+d_{13}\wedge d_{24}\wedge d_{34}\wedge d_{45}$
26	$(0,-2,2,-1) = 3\omega_1 - 3\alpha_1 - 3\alpha_2 - \alpha_3 - \alpha_4$	$d_{23}\wedge d_{34}\wedge d_{15}\wedge d_{35}+d_{13}\wedge d_{23}\wedge d_{35}\wedge d_{45}-d_{13}\wedge d_{34}\wedge d_{25}\wedge d_{35}$
27	$(-1,1,-1,0) = 3\omega_1 - 3\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$	$d_{23}\wedge d_{24}\wedge d_{15}\wedge d_{45}+d_{24}\wedge d_{34}\wedge d_{15}\wedge d_{25}-d_{23}\wedge d_{14}\wedge d_{25}\wedge d_{45}+$ $d_{12}\wedge d_{24}\wedge d_{35}\wedge d_{45}-d_{12}\wedge d_{34}\wedge d_{25}\wedge d_{45}-d_{14}\wedge d_{24}\wedge d_{25}\wedge d_{35}$
28	$(1,0,0,-2) = 3\omega_1 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$	$d_{14}\wedge d_{15}\wedge d_{25}\wedge d_{35}-d_{13}\wedge d_{15}\wedge d_{25}\wedge d_{45}+d_{12}\wedge d_{15}\wedge d_{35}\wedge d_{45}$
29	$(0,-1,0,0) = 3\omega_1 - 3\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$	$d_{24}\wedge d_{34}\wedge d_{15}\wedge d_{35}-d_{14}\wedge d_{34}\wedge d_{25}\wedge d_{35}+d_{23}\wedge d_{34}\wedge d_{15}\wedge d_{45}-$ $d_{23}\wedge d_{14}\wedge d_{35}\wedge d_{45}-d_{13}\wedge d_{34}\wedge d_{25}\wedge d_{45}+d_{13}\wedge d_{24}\wedge d_{35}\wedge d_{45}$
30	$(-1,1,-1,0) = 3\omega_1 - 3\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$	$-d_{23}\wedge d_{15}\wedge d_{25}\wedge d_{45}+d_{24}\wedge d_{15}\wedge d_{25}\wedge d_{35}+d_{12}\wedge d_{25}\wedge d_{35}\wedge d_{45}$
31	$(0,0,-3,3) = 3\omega_1 - 3\alpha_1 - 3\alpha_2 - 3\alpha_3$	$d_{14} \wedge d_{24} \wedge d_{34} \wedge d_{45}$
32	$(0,0,-2,1) = 3\omega_1 - 3\alpha_1 - 3\alpha_2 - 3\alpha_3 - \alpha_4$	$d_{24}\wedge d_{34}\wedge d_{15}\wedge d_{45}-d_{14}\wedge d_{34}\wedge d_{25}\wedge d_{45}+d_{14}\wedge d_{24}\wedge d_{35}\wedge d_{45}$
33	$(0,-1,1,-2) = 3\omega_1 - 3\alpha_1 - 3\alpha_2 - 2\alpha_3 - 2\alpha_4$	$-d_{23}\wedge d_{15}\wedge d_{35}\wedge d_{45}+d_{34}\wedge d_{15}\wedge d_{25}\wedge d_{35}+d_{13}\wedge d_{25}\wedge d_{35}\wedge d_{45}$
34	$(0,0,-1,-1) = 3\omega_1 - 3\alpha_1 - 3\alpha_2 - 3\alpha_3 - 2\alpha_4$	$-d_{24}\wedge d_{15}\wedge d_{35}\wedge d_{45}+d_{14}\wedge d_{25}\wedge d_{35}\wedge d_{45}+d_{34}\wedge d_{15}\wedge d_{25}\wedge d_{45}$
35	$(0,0,0,-3) = 3\omega_1 - 3\alpha_1 - 3\alpha_2 - 3\alpha_3 - 3\alpha_4$	$d_{15} \wedge d_{25} \wedge d_{35} \wedge d_{45}$

Table 9: **Weights for $\mathbf{sl}_5$ module** $V(\omega_1 + \omega_2)$

i	$\overrightarrow{w}_i^{\omega_2}$	i	$\overrightarrow{w}_i^{\omega_2}$
1	$(1,1,0,0) = \omega_1 + \omega_2$	16	$(1,0,-2,2) = \omega_1 + \omega_2 - \alpha_1 - 2\alpha_2 - 2\alpha_3$
2	$(-1,2,0,0) = \omega_1 + \omega_2 - \alpha_1$	17	$(1,-1,1,-1) = \omega_1 + \omega_2 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4$
3	$(2,-1,1,0) = \omega_1 + \omega_2 - \alpha_2$	18	$(0,-2,1,1) = \omega_1 + \omega_2 - 2\alpha_1 - 3\alpha_2 - \alpha_3$
4	$(0,0,1,0) = \omega_1 + \omega_2 - \alpha_1 - \alpha_2$	19	$(-1,1,-2,2) = \omega_1 + \omega_2 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3$
5	$(2,0,-1,1) = \omega_1 + \omega_2 - \alpha_2 - \alpha_3$	20	$(-1,0,1,-1) = \omega_1 + \omega_2 - 2\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4$
6	$(-2,1,1,0) = \omega_1 + \omega_2 - 2\alpha_1 - \alpha_2$	21	$(1,0,-1,0) = \omega_1 + \omega_2 - \alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$
7	$(1,-2,2,0) = \omega_1 + \omega_2 - \alpha_1 - 2\alpha_2$	22	$(0,-1,-1,2) = \omega_1 + \omega_2 - 2\alpha_1 - 3\alpha_2 - 2\alpha_3$
8	$(0,1,-1,1) = \omega_1 + \omega_2 - \alpha_1 - \alpha_2 - \alpha_3$	23	$(0,-2,2,-1) = \omega_1 + \omega_2 - 2\alpha_1 - 3\alpha_2 - \alpha_3 - \alpha_4$
9	$(2,0,0,-1) = \omega_1 + \omega_2 - \alpha_2 - \alpha_3 - \alpha_4$	24	$(-1,1,-1,0) = \omega_1 + \omega_2 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$
10	$(-1,-1,2,0) = \omega_1 + \omega_2 - 2\alpha_1 - 2\alpha_2$	25	$(1,0,0,-2) = \omega_1 + \omega_2 - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$
11	$(-2,2,-1,1) = \omega_1 + \omega_2 - 2\alpha_1 - \alpha_2 - \alpha_3$	26	$(0,-1,0,0) = \omega_1 + \omega_2 - 2\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
12	$(1,-1,0,1) = \omega_1 + \omega_2 - \alpha_1 - 2\alpha_2 - \alpha_3$	27	$(-1,1,0,-2) = \omega_1 + \omega_2 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$
13	$(0,1,0,-1) = \omega_1 + \omega_2 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$	28	$(0,0,-2,1) = \omega_1 + \omega_2 - 2\alpha_1 - 3\alpha_2 - 3\alpha_3 - \alpha_4$
14	$(-1,0,0,1) = \omega_1 + \omega_2 - 2\alpha_1 - 2\alpha_2 - \alpha_3$	29	$(0,-1,1,-2) = \omega_1 + \omega_2 - 2\alpha_1 - 3\alpha_2 - 2\alpha_3 - 2\alpha_4$
15	$(-2,2,0,-1) = \omega_1 + \omega_2 - 2\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$	30	$(0,0,-1,-1) = \omega_1 + \omega_2 - 2\alpha_1 - 3\alpha_2 - 3\alpha_3 - 2\alpha_4$